Chapter 2 – Limits and Continuity

2.1 Rates of Change and Limits

recall: A moving body's average speed during an interval of time is found by dividing the distance covered by the elapsed time. The unit of measure is length per unit of time (i.e. ft per sec, km per hour, etc…)

ex. A rock breaks loose from the top of a tall cliff. What is the average speed during the first 2 seconds of fall?

Experiments show that dense objects dropped from rest (not moving) to falling freely near the surface of the Earth will fall $\boxed{y=16t^2}$ feet in the first t seconds. Therefore, the average speed of the rock over any given interval is the distance traveled (Δy), divided by the length of time (Δt)

$$\frac{\Delta y}{\Delta t}=\frac{y_2-y_1}{t_2-t_1}=\frac{16(2)^2-16(0)^2}{2-0}=\frac{64-0}{2}=32\,\tfrac{ft}{\sec}$$

Suppose we wanted to know the exact speed of the rock AT TIME t = 2 seconds (NOT THE AVERAGE).

Suppose we let h represent the amount of time after 2 seconds. Then, the later time would be written as $t = 2 + h$. Therefore, the average speed would be:

$$\frac{\Delta y}{\Delta t}=\frac{16(2+h)^2-16(2)^2}{2+h-2}=\frac{16(2+h)^2-16(2)^2}{h}$$

To find the exact speed at t=2, we would need to make h as small as possible without being 0. Why?

We could use some values of h and see what happens as h gets smaller and smaller:

Length of time interval h	Average Speed for the interval $\frac{\Delta y}{\Delta t}=\frac{16(2+h)^2-16(2)^2}{h}$
1	
0.1	
0.01	
0.001	
0.0001	
0.00001	

As you can see, as h gets closer and closer to 0, the average speed is getting closer to _______ $\tfrac{ft}{\sec}$

We refer to this value as a limiting value as h approaches 0.

We can confirm this using algebra:

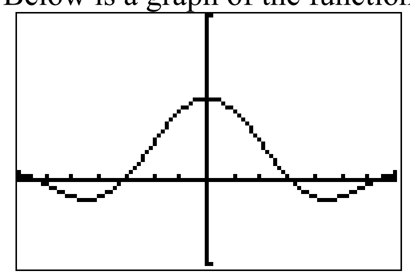
Simplify: $\dfrac{16(2+h)^2-16(2)^2}{h}$

We can say that the original expression $\dfrac{16(2+h)^2-16(2)^2}{h}$ and the result are equivalent because we say that $h \neq 0$, it simply gets closer and closer to 0, but never gets to 0, so the resulting expression is equivalent to the original.

Ex. Given $f(x)=\dfrac{\sin x}{x}$

What is the domain?____________

Below is a graph of the function



$-7 < x < 7 \quad -1 < y < 1$

Here is a table of values for *f(x)*

X	Y1
-.3	.98507
-.2	.99335
-.1	.99833
0	ERROR
.1	.99833
.2	.99335
.3	.98507

X=-.3

as *x* approaches 0, *f(x)* approaches 1

What does the graph show at $x = 0$?

We cannot eliminate the *x* like we did *h* in the previous example using algebra (for now at least).

<u>Def</u>**: Limit**

Let *c* and *L* be real numbers. The function ***f(x)* has limit *L* as *x* approaches *c*** if, given any positive number ε, there is a positive number δ such that for all *x*:

$$0 < |x-c| < \delta \Rightarrow |f(x)-L| < \varepsilon$$

We write:

$$\lim_{x \to c} f(x) = L$$

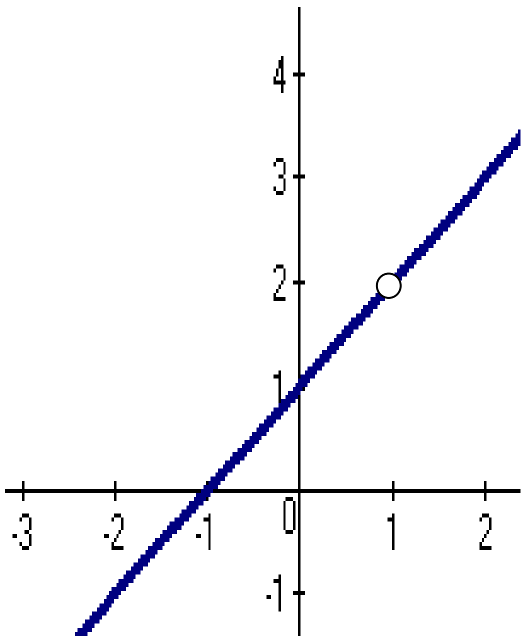
This is read:
"The limit of *f* of *x* as *x* approaches *c* is *L*"

δ – Greek Letter "delta"
ε – Greek Letter "epsilon"

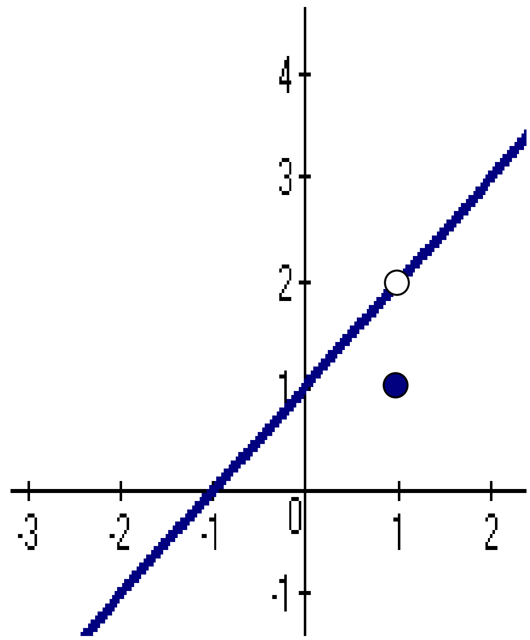
The notation $\lim_{x \to c} f(x) = L$ means that the values of *f(x)* of the function *f* approach or equal *L* as the values of *x* approach (but **do not equal**) *c*.

Going back to an example: $\lim_{h \to 0} \frac{16(2+h)^2 - 16(2)^2}{h} = \lim_{h \to 0}(64 + 16h) = 64$

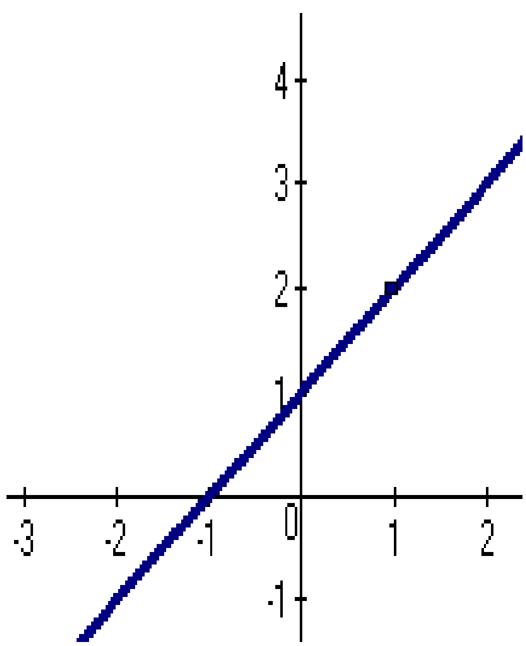
Using the next example and the chart, we get $\lim_{x \to 0} \frac{\sin x}{x} = 1$.



$f(x) = \frac{x^2 - 1}{x - 1}$



$g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, x \neq 1 \\ 1, x = 1 \end{cases}$



$h(x) = x + 1$

The three graphs here show that the existence of a limit as $x \to c$ never depends on how the function may or may not be defined at *c*.

As $x \to 1$:

f approaches 2 even though *f* is undefined at 1

g approaches 2 even though *f*(1) = 1

h approaches 2 (the only one of the three functions whose limit at *x* = 1 is the same as its value at 1

Properties of Limits: If *L, M,* and *k* are real numbers and

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$

1. $\lim_{x \to c}(k) = k$ – The limit of a constant is equal to the constant.
2. $\lim_{x \to c}(x) = c$ – The limit of the identity function ($y = x$) at $x = c$
3. **Sum Rule:** $\lim_{x \to c}(f(x) + g(x)) = L + M$ – The limit of a sum is the sum of their limits
4. **Difference Rule:** $\lim_{x \to c}(f(x) - g(x)) = L - M$ – The limit of a difference is the difference of their limits
5. **Product Rule:** $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$ – The limit of a product is the product of their limits
6. **Constant Product Rule:** $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$ – The limit of a constant times a function is the constant times the limit of the function
7. **Quotient Rule:** $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, M \neq 0$ – The limit of a quotient is the quotient of their limits
8. **Power Rule:** If r and s are integers, $s \neq 0$, then

   $\lim_{x \to c}(f(x))^{\frac{r}{s}} = L^{\frac{r}{s}}$ provided that $L^{\frac{r}{s}}$ is a real number

   The limit of a rational power of a function is the power of the limit of the function, provided the latter is a real number.

Examples

HW: Pg. 62–63 #1-20, 21-29 odd

2.1 Rates of Change and Limits (con't)

Limits of Polynomial and Rational Functions

<u>Theorem</u>: *Polynomial and Rational Functions*

1. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is any polynomial function and c is any real number, then

$$\lim_{x \to c} f(x) = f(c) = a_n c^n + a_{n-1}c^{n-1} + \ldots + a_1 c + a_0$$

2. If *f(x)* and *g(x)* are polynomials and *c* is any real number, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f(c)}{g(c)}, \text{ provided that } g(c) \neq 0$$

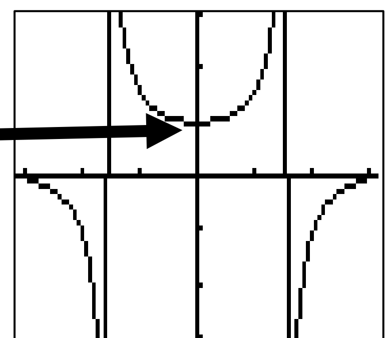
ex.

(a) $\lim_{x \to 3} [x^2(2-x)] =$

(b) $\lim_{x \to 2} \frac{x^2 + 2x + 4}{x+2} =$

ex. Determine $\lim_{x \to 0} \frac{\tan x}{x}$

To the right is the graph of the function



x ε [–π,π] y ε [–3, 3]

Algebraically:

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{1}{x} \tan x$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \frac{\sin x}{\cos x}$$

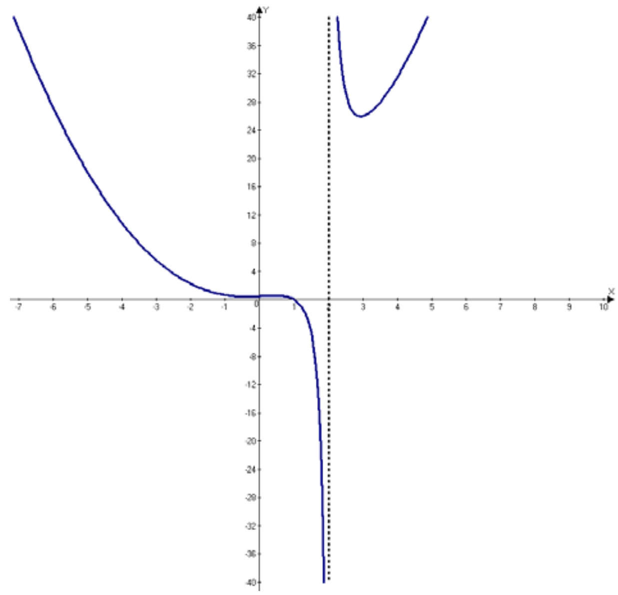
$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{\cos x}$$

$$= 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = \boxed{1}$$

ex. Show that $\lim_{x\to 2}\frac{x^3-1}{x-2}$ does not exist.

Graphically, the function $f(x)=\frac{x^3-1}{x-2}$ looks like this:



Note: since the denominator is $x-2$, when we substitute 2 in, it would make the fraction undefined.

Using the graph:

- Looking at the graph, there does not appear to a particular value that the graph is approaching.
  - as we approach 2 from the left side(1.9, 1.99, 1.999, …), the graph appears to move more negative (approaches $-\infty$).
    - We write a limit as we approach c from the left side:

      $\lim_{x\to c^-} f(x)$
    - We call this a ***left-handed limit***
  - as we approach 2 from the right side (i.e. 2.1, 2.01, 2.001, …), the graph gets larger and larger (approaches $+\infty$).
    - We write a limit as we approach c from the right hang side:

      $\lim_{x\to c^+} f(x)$
    - We call this a ***right-handed limit***

Theorem: *One-sided and Two-sided Limits*

A function *f(x)* has a limit as x approaches c iff ("*if and only if*") the right-hand and left-hand limits at c exist and are equal.

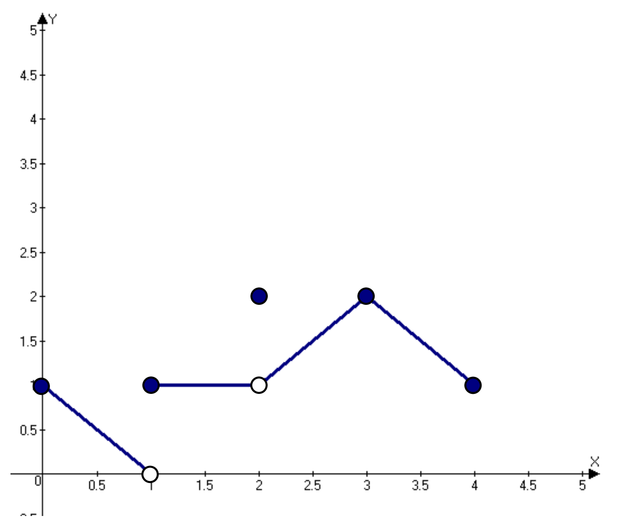
Symbolically: $\lim_{x\to c} f(x)=L \Leftrightarrow \lim_{x\to c^-} f(x)=L$ and $\lim_{x\to c^+} f(x)=L$

Therefore, in order for $\lim_{x\to c} f(x)$ to exist, it must have equal limits from the left and right sides.

Ex. Use the graph to the right to answer the following questions:

The function is defined:

$$f(x) = \begin{cases} -x+1, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \\ x-1, & 2 < x \le 3 \\ -x+5, & 3 < x \le 4 \end{cases}$$



(a) $\lim_{x \to 1} f(x)$

(b) $\lim_{x \to 2^-} f(x)$

(c) $\lim_{x \to 2^+} f(x)$

(d) $\lim_{x \to 2} f(x)$

(e) $\lim_{x \to 3} f(x)$

Sandwich Theorem:

- If we can not find a limit directly, we can find it indirectly using the Sandwich Theorem
  - If a function *f* has values that are between two other functions *g* and *h* (sandwiched between) and *g* and *h* have the same limit as $x \to c$, then *f* has the same limit as well.

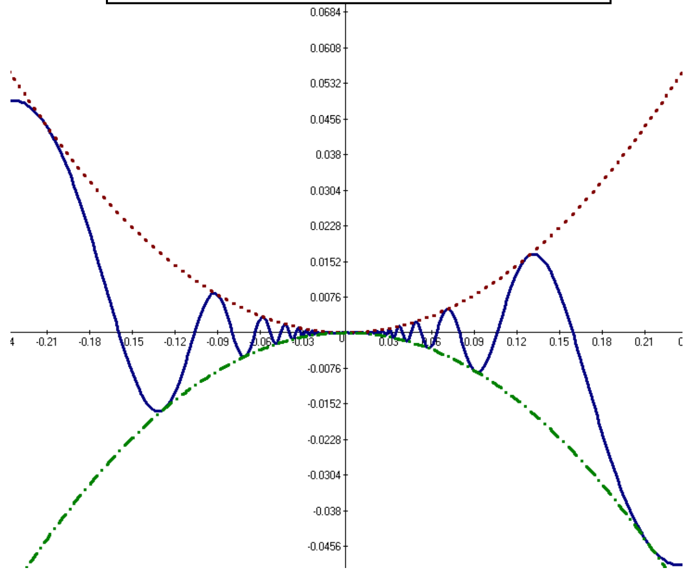
Theorem: The *Sandwich Theorem*

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about *c*, and $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$, then

$$\lim_{x \to c} f(x) = L$$

ex. Show $\lim_{x\to 0}[x^2 \sin(\frac{1}{x})] = 0$

x ε (–0.3, 0.3) y ε (–0.03,0.03)



The graph to the right depict the graph of $f(x) = x^2 \sin(\frac{1}{x})$ (the solid graph)

The dotted curve is $y=x^2$ and the dashed curve is $y=-x^2$. Notice that that $f(x)$ is between the other two curves throughout the interval.

We also know that the value of the sine is always between –1 and 1, so

$$\left|x^2 \sin(\tfrac{1}{x})\right| = \left|x^2\right| \cdot \left|\sin(\tfrac{1}{x})\right| \le \left|x^2\right| \cdot 1$$

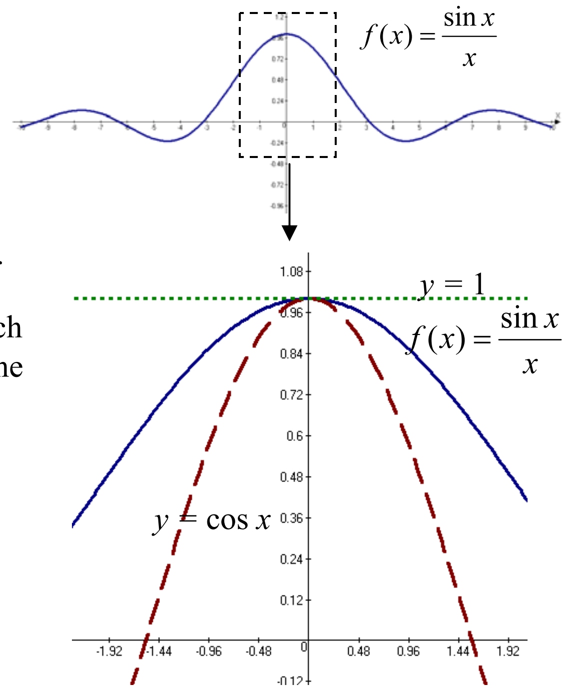
$$\left|x^2\right| \cdot \left|\sin(\tfrac{1}{x})\right| \le x^2$$

Therefore: $-x^2 \le \left|x^2\right| \cdot \left|\sin(\tfrac{1}{x})\right| \le x^2$

Now, $\lim_{x\to 0} -x^2 = 0$ and $\lim_{x\to 0} x^2 = 0$.

By the Sandwich Theorem, $\lim_{x\to 0}[x^2 \sin(\frac{1}{x})] = 0$

Proof of $\lim_{x\to 0} \frac{\sin x}{x} = 1$



The solid graph to the right shows $f(x) = \frac{\sin x}{x}$. Two other functions $g(x) = \cos x$ and $h(x) = 1$ are two functions in which $f(x)$ is between. Since $\lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} 1 = 1$, then by the Sandwich Theorem, it shows that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Homework: Pg. 62-64 #22-30even, 31, 32, 43, 44

2.1c More with Limits

<u>Theorem</u>: The Limit of a Function Involving a Radical

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

<u>Theorem</u>: The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f(L)$$

ex. Find $\lim_{x \to 0} \sqrt{x^2 + 4}$

$\sqrt{x^2 + 4}$ is a composite of two functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 4$, therefore $\lim_{x \to 0} \sqrt{x^2 + 4}$ can be found in this method:

$$\lim_{x \to 0}(x^2 + 4) = 0^2 + 4 = 4$$

$$\lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$$

<u>Theorem</u>: Limits of Trig Functions

Let c be a real number in the domain of the given trigonometric function

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \csc x = \csc c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \cot x = \cot c$

ex. Evaluate

1. $\lim_{x \to 0} \tan x$

2. $\lim_{x \to \pi} (x \cos x)$

3. $\lim_{x \to 0} \sin^2 x$

Strategies for Finding Limits

So far most limits you have done can be done with direct substitution, but suppose you end up with something rather odd…..

Ex. Find $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$

By substitution, this limit turns out to be $\dfrac{0}{0}$!!!

This is a tell-tale sign that something can be done.

> **Theorem**: Functions That Agree at All But One Point
>
> Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and
>
> $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

This theorem allows you to factor and cancel.

Ex. Find $\lim\limits_{x\to 1}\dfrac{x^3-1}{x-1}$

g(x)

$$\lim_{x\to 1}\frac{x^3-1}{x-1} = \lim_{x\to 1}\frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = \lim_{x\to 1}(x^2+x+1) = 1^2+1+1=3$$

A Strategy to find $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$:

- First attempt a straight substitution into the expression.
- If that results in a "$\frac{0}{0}$", try and see if the numerator and denominator will factor using a common factor ($x - c$).
    - If it works out that there is, then you can cancel the common factors and use the theorem above.
    - If it does not work out, then it is a good bet that there is no limit (does not exist)
- If a radical is involved in the expression, *rationalize* the expression
- Careful with trig functions
- Use a graph or a table as a last resort.

<u>Rationalization Technique</u> – Use with RADICALS

> <u>Def</u>: Conjugate of $a + b$
>
> The conjugate of $a + b$ is $a - b$
>
> The product of a binomial $a + b$ and its conjugate $a - b$ is
>
> $$a^2 - b^2$$
>
> Very useful to get rid of radicals!

ex. Find $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

Solution:

Direct Substitution: $\frac{0}{0}$ And it is not factorable.

But a radical is involved.

Multiply the top and bottom by the conjugate of the top ("1" in disguise)

$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\lim_{x\to 0}\frac{(x+1)-1}{x(\sqrt{x+1}+1)}=\lim_{x\to 0}\frac{\cancel{x}}{\cancel{x}(\sqrt{x+1}+1)}=\lim_{x\to 0}\frac{1}{\sqrt{x+1}+1}=\frac{1}{1+1}=\frac{1}{2}$$

ex. Find

1. $\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$

2. $\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3}$

3. $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$

4. $\lim_{x\to 0}\frac{\sin 4x}{x}$

Homework: Limit Worksheet

2.2 Limits Involving Infinity

- The symbol for *infinity* (∞) does not represent a real number
  - We use ∞ to describe the behavior of a function
    - Used when the domain or range "outgrow" all finite bounds
    - $\lim_{x\to\infty} f(x)$ means "the limit of *f* as *x* approaches infinity
      - as *x* moves increasingly far to the right on the number line
    - as "$x\to -\infty$" means as *x* moves increasingly far to the left on the number line.
  - In these cases, the limit may or may not exist.

Ex. Find 1. $\lim_{x\to\infty} \frac{1}{x}$

2. $\lim_{x\to-\infty} \frac{1}{x}$

Since both of these limits result in the same value, we say that the line $y = 0$ is a horizontal asymptote of a function.

> Def: The line $y = b$ is a **horizontal asymptote** of the graph of $y = f(x)$ if either
> $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$

ex. Find the horizontal asymptote for $f(x)$

1. $f(x) = 2 + \frac{1}{x}$

2. $f(x) = \frac{x}{\sqrt{x^2+1}}$ (hint: make a table)

> Def: The line $x = a$ is a **vertical asymptote** of the graph of $y = f(x)$ if either
> $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$

To find a vertical asymptote:

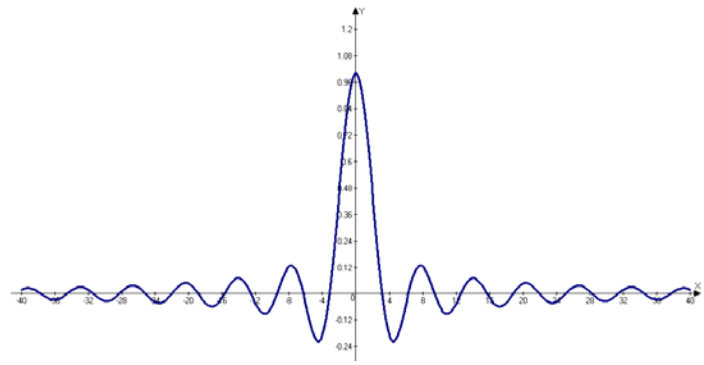
1. Determine the domain (look for values *x* can not be)
2. Take the limit as *x* approaches these values and if the limit is $\pm\infty$, then it is an asymptote.

Ex. Find $\lim_{x\to\infty} \frac{\sin x}{x}$

Here is a table of values

X	Y1	
100	-.0051	
200	-.0044	
300	-.0033	
400	-.0021	
500	-9E-4	
600	7.4E-5	
700	7.8E-4	

X=700



The graph of the function $y = \frac{\sin x}{x}$

As you can see, as $x \to \infty$, the value of y gets closer and closer to 0.

<u>Analytical Solution</u>:

We know that $-1 \le \sin x \le 1$, so for $x > 0$, we have:

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

By the Sandwich Theorem: $0 = \lim_{x\to\infty}\left(-\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\sin x}{x} = \lim_{x\to\infty}\left(\frac{1}{x}\right)$. Since $\frac{\sin x}{x}$ is an even function, we can also conclude that $\lim_{x\to-\infty}\frac{\sin x}{x} = 0$

<u>Theorem</u>: Properties of Limits as $x \to \pm\infty$

If L, M, and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \text{ and } \lim_{x\to\pm\infty} g(x) = M \text{, then}$$

1. **<u>Sum Rule</u>:** $\lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$

2. **<u>Difference Rule</u>:** $\lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$

3. **<u>Product Rule</u>:** $\lim_{x\to\pm\infty}(f(x) \cdot g(x)) = L \cdot M$

4. **<u>Constant Product Rule</u>:** $\lim_{x\to\pm\infty}(k \cdot f(x)) = k \cdot L$

5. **<u>Quotient Rule</u>:** $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = \frac{L}{M}$, $M \ne 0$

6. **<u>Power Rule</u>:** If r and s are integers, $s \ne 0$, then

$$\lim_{x\to\pm\infty}(f(x))^{\frac{r}{s}} = L^{\frac{r}{s}} \text{ provided that } L^{\frac{r}{s}} \text{ is a real number}$$

ex. Find $\lim_{x\to\infty} \frac{5x + \sin x}{x}$

Strategy: If *f(x)* and *g(x)* are polynomial functions, then

1. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$, if the *degree* of $f(x)$ < *degree* of $g(x)$
2. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, if the *degree* of $f(x)$ > *degree* of $g(x)$
3. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$, if the degrees of f and g are the same and a and b are the leading coefficients

<u>Alternate Approach</u>: Divide every term by the variable to the HIGHEST exponent and cancel accordingly. Any term with a variable in the bottom cancels to 0 (goes away).

ex. Find

1. $\lim_{x\to\infty} \frac{4x^3 - 3x + 4}{3x^5 - 3x + 1} =$
2. $\lim_{x\to\infty} \frac{6x^2 - 3x + 7}{2x^2 - 8} =$
3. $\lim_{x\to\infty} \frac{x^3}{2x} =$
4. $\lim_{x\to\infty} \frac{4x - 5x^2}{10x^2 - 1} =$

End Behavior Models

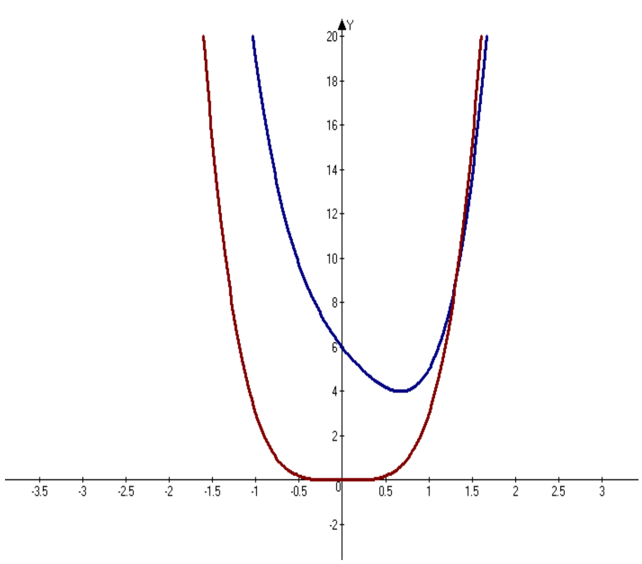
- For large values of x (positive or negative), we can sometimes model the behavior of a complicated function by a simpler one.

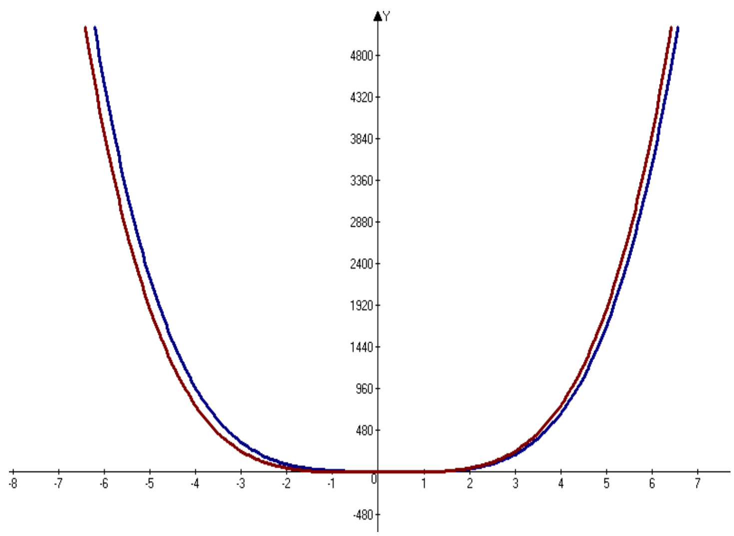
> Def: The function g is
>
> - a **right end behavior model** for f if and only if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$
> - a **left end behavior model** for f if and only if $\lim_{x\to-\infty}\frac{f(x)}{g(x)}=1$

A function's right and left behavior model need not be the same function

Ex. Suppose $f(x)=3x^4-2x^3-5x+6$. $g(x)=3x^4$, though considerably different from *f(x)* with smaller values of x, are virtually identical for large values of x.

Look at the graphs below of the two functions





$x\ \varepsilon\ [-3,3]$ $y\ \varepsilon\ (-3,20)$ $x\ \varepsilon\ [-7,7]$ $y\ \varepsilon\ (-480,5000)$

Both of these graphs represent *f(x)* and *g(x)*. Note the differences with the graph on the left, but they look almost identical when expanded out to the graph on the right.

Analytically…

$$\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}=\lim_{x\to\pm\infty}\frac{3x^4-2x^3+3x^2-5x+6}{3x^4}=\lim_{x\to\pm\infty}\left(1-\underbrace{\frac{2}{3x}}_{0}+\underbrace{\frac{1}{x^2}}_{0}-\underbrace{\frac{5}{3x^3}}_{0}+\underbrace{\frac{2}{x^4}}_{0}\right)=\boxed{1}$$

<u>To find End Behavior Models</u>: For $h(x)=\dfrac{f(x)}{g(x)}$

1. Write the leading term for *f* and *g* as a fraction
2. Simplify the fraction for the end behavior

Ex. Find the end behavior model for

(a) $f(x)=\dfrac{4x^5+3x^3-2x+6}{2x^2-6x+8}$

(b) $g(x)=\dfrac{2x^3-x^2+x-1}{5x^3+x^2+x-5}$

ex. Does the graph of $f(x)=\dfrac{4x^2-3x+5}{2x^3+x-1}$ have a horizontal asymptote? If so, what is it?

Ex. Find $\lim_{x\to\infty}\sin(\frac{1}{x})$

Homework: Pg. 71–73 Exercises #1-7 odd, 10, 12, 13, 14, 17-20, 23-28

2.3 Continuity

- In mathematics, the term *continuous* has much the same meaning as it has in everyday usage.
  - To say a function is continuous at $x = c$ mean that there is not interruption in the graph of f at c.
    - The graph is *unbroken* at c and there are no holes, jumps, gaps, etc…
    - You can think that *a function is continuous on an open interval if its graph can be drawn with a pencil without lifting it off the paper.*
    - Below are examples of functions that are not continuous

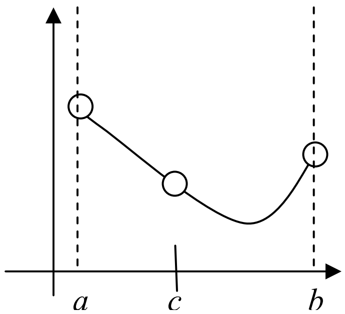


Figure A

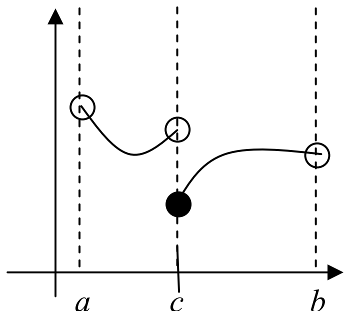


Figure B

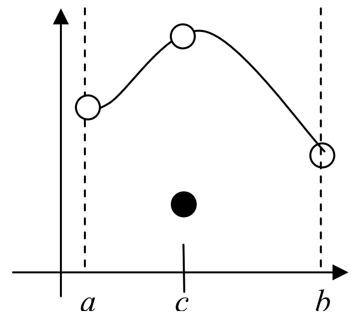


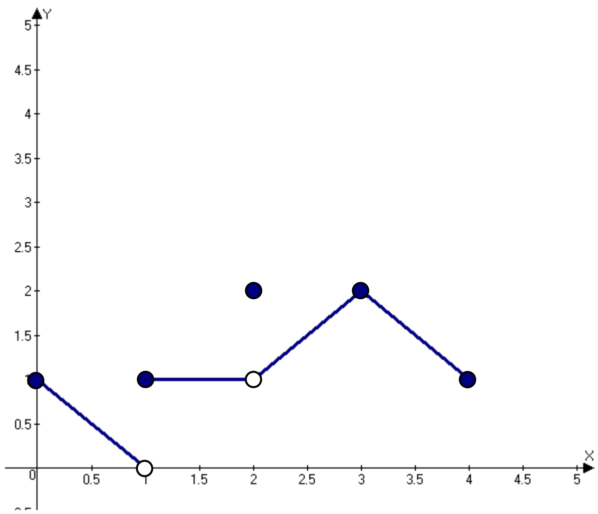
Figure C

- Figure A has a hole in it: $f(c)$ does not exist
- Figure B has a jump: $\lim_{x \to c} f(x)$ does not exist
- Figure C has a gap: $\lim_{x \to c} f(x) \neq f(c)$

> <u>Def</u>: ***Continuity at a Point***
>
> A function f is <u>*continuous at a point c*</u> if the following three conditions are met:
>
> 1. $f(c)$ exists
> 2. $\lim_{x \to c} f(x)$ exists
> 3. $\lim_{x \to c} f(x) = f(c)$
>
> *Continuity on an Open Interval:* A function is continuous on an open interval (a,b) if it is continuous at each point in the interval.
>
> A function that is continuous on the entire real number line $(-\infty,\infty)$ is <u>***everywhere continuous***</u>.
>
> ***Continuity at an Endpoint***
>
> A function $y = f(x)$ is ***continuous at a left endpoint a*** or ***continuous at a right endpoint b*** of its domain if
>
> $\lim_{x \to a^+} f(x) = f(a)$ or $\lim_{x \to b^-} f(x) = f(b)$ respectively

- If a function is not continuous at a point $x = c$, then it is said to be <u>*discontinuous at $x = c$*</u>

Ex. Use the graph below to answer the following:



For what domain is the function continuous?

At what value(s) of x is the function discontinuous?

- Discontinuous points fall into two categories:
  - Removable – has a limit at $x=c$
    - f can be made continuous by appropriately defining or redefining $f(c)$.
    - In the graph above, if we redefine f at $x=2$, we can make the function continuous at $x = 2$
  - non-removable
    - f can not be made continuous at $x=1$, since it would requiring redefining the function at several points

ex. Discuss the continuity of each of the following

1. $f(x) = \dfrac{1}{x}$

2. $g(x) = \dfrac{x^2 - 1}{x - 1}$

3. $h(x) = \begin{cases} x+1, x \le 0 \\ x^2 + 1, x > 0 \end{cases}$

4. $y = \sin x$

Def: ***The Greatest Integer Function (a.k.a. "The Birthday Function")***

The Greatest Integer Function, $y = \lfloor x \rfloor$ has the properties such that for every non-integer value of *x*, *y* equals the largest integer less than or equal to *x*.

- Basically it always rounds down to the previous integer.
- On many graphing calculators and computers, this is the int(*x*) function

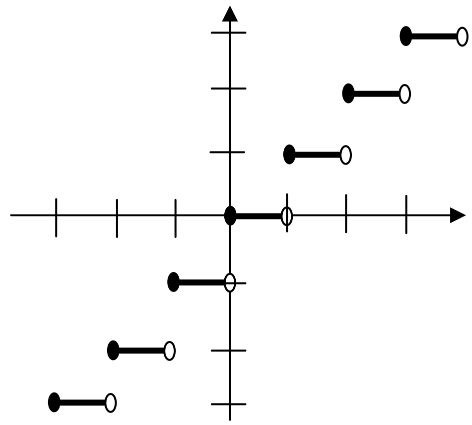
Ex.

$\lfloor 4.5 \rfloor = 4$

$\lfloor -1.4 \rfloor = -2$

$\lfloor 5.999 \rfloor = 5$

$\lfloor 3 \rfloor = 3$



- A graph of the function is shown to the right
- For what values is the function discontinuous?

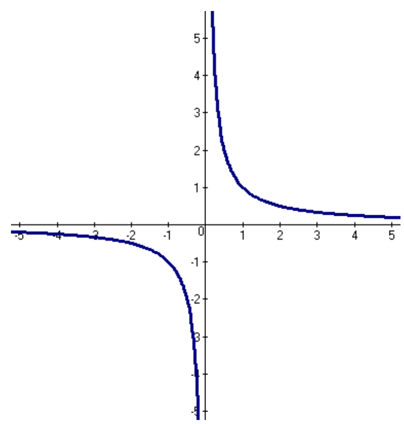
Prove it algebraically at one of the values:

Def: ***Continuous Functions***

A function is a *continuous function* if and only if it is continuous at every point of its domain.

Note: A continuous function does not need to be continuous on every interval.

Ex. $y = \dfrac{1}{x}$



As you can , the function is not continuous at $x = 0$, but if you specify the domain such that 0 is not if the domain, then the function would be a continuous function (i.e. [1,10])

- What functions are continuous (or where are they discontinuous?)?
  - Polynomial functions are continuous at every real number *c* because $\lim_{x \to c} f(x) = f(c)$
  - Rational functions, $f(x) = \dfrac{g(x)}{h(x)}$ are continuous at every point of their domain
    - They are discontinuous at every point where $h(x) = 0$
  - The absolute value function $y = |x|$ is continuous at every real number
  - The exponential functions, logarithmic functions, trig functions, and radical functions like $y = \sqrt[n]{x}$ are continuous at every point of their domains

Theorem: *Properties of Continuous Functions*

If the functions f and g are continuous at $x = c$, then the following combinations are also continuous at $x = c$:

1. *Sum:* $f + g$
2. *Difference:* $f - g$
3. *Products* $f \cdot g$
4. *Constant Multiples:* $k \cdot f$ for any number k
5. *Quotients:* f/g provided $g(c) \neq 0$

Composite Functions

- If f and g are continuous functions, then $(f \circ g)(x)$ or $f(g(x))$ is also a composite function.

Ex. $y = \sin(x^2)$ is a continuous function because $y = \sin x$ and $y = x^2$ are continuous functions

Ex. Is the function $y = |\cos x|$ a continuous function? Why or why not?

Ex. Show that $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ is continuous

Solution: Break this function into two functions:

$$y = |x| \quad \text{and} \quad y = \frac{x \sin x}{x^2 + 2}$$

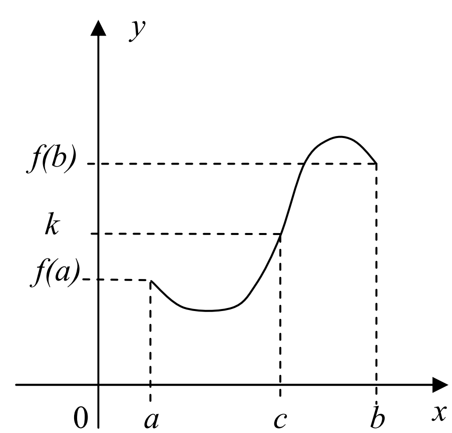
The absolute value function is continuous (shown earlier)

The other function is continuous according to #5 of the above theorem ($x\sin x$ is continuous and $x^2 + 2$ is continuous, therefore their quotient is.

Therefore $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ is continuous

<u>Theorem</u>: Intermediate Value Theorem for Continuous Functions

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

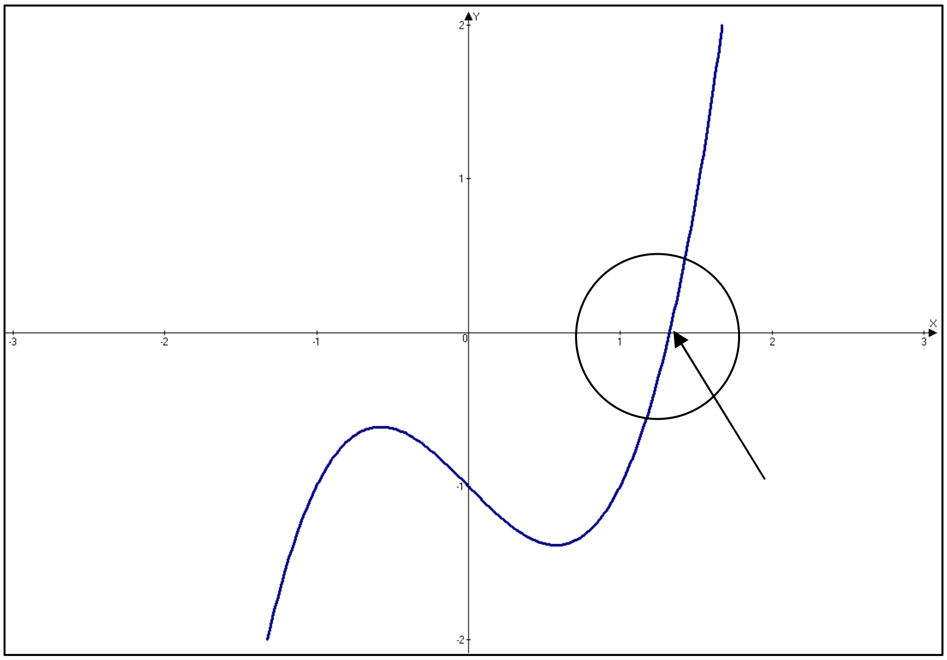


Ex. Is there a real number exactly 1 less than its value cubed?

As an equation: $x = x^3 - 1$
Rewriting the equation: $x^3 - x - 1 = 0$

So we are looking for the zero value for the function $f(x) = x^3 - x - 1$

Looking at its graph, we can see that the curve crosses the x-axis between $x = 1$ and $x = 2$. Therefore there is a point between 1 and 2 where $f(c) = 0$, hence showing that there is a real value 1 less than its cube.



Homework: Pg 80 – 81 #1–9 odd, 11-16, 19-24 (answer part a only), 35 – 40

2.4 Rates of Change and Tangent Lines

recall: The average rate of change of a quantity over a period of time is the amount of change divided by the time it takes.

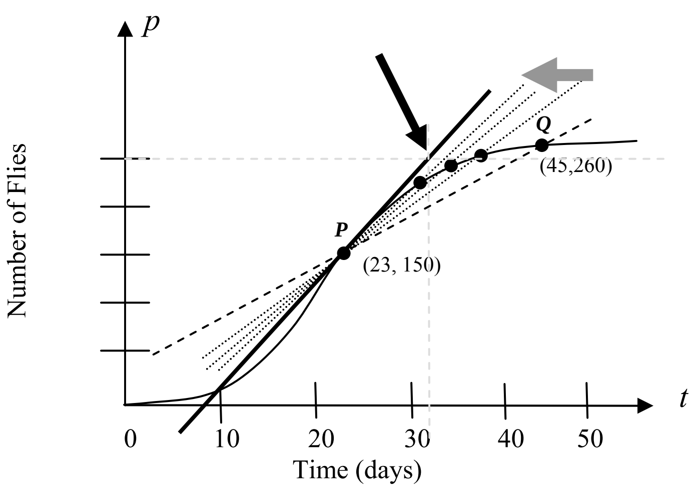
Ex. Find the average rate of change of $f(x) = x^3 - x$ over the interval [–3, 3].

$$\frac{\Delta f}{\Delta t} = \frac{f(3) - f(-3)}{3 - -3} = \frac{24 - (-24)}{6} = \frac{48}{6} = 8$$

Ex. Suppose an experiment counted the number of fruit flies over a period of time. 23 days into the experiment, there were 150 fruit flies and after 45 days, there were 260 fruit flies. What was the average rate of change?

$$\frac{\Delta f}{\Delta t} = \frac{f(45) - f(23)}{45 - 23} = \frac{260 - 150}{45 - 23} = \frac{110}{22} = 5$$

Now, suppose the fruit fly counting problem resulted in a curved function below.



The two points when graphed form a secant.

Def: A *secant* is a line that intersects a curve in at least 2 points.

The slope of the secant here is $\frac{\Delta p}{\Delta t}$ which is the average rate of change of the number of flies.

Suppose we wanted to know the exact rate of change on day 23. If we backed the 2nd point closer and closer to day 23 and found the slope, we would have a good indicator on the rate of change on day 23 (also called the instantaneous rate of change).

To the right is a table of the 4 secants as we go from 45 days back to 23 days:

Q	Slope of *PQ*
(45,260)	(260-150)/(45-23) = 5
(38,250)	(250-150)/(38-23)= 6.667
(33,240)	(240-150)/(33-23) = 9
(31,225)	(225-150)/(21-23) = 9.375

Eventually, the point will end up at *P*. The line that intersect the curve only at P would be called the *tangent* (the solid line with arrow pointing at it). The slope of this line would be the instantaneous rate of change of the curve at *P*. The slope is found using two points *A*(9,0) and *B*(32,250)

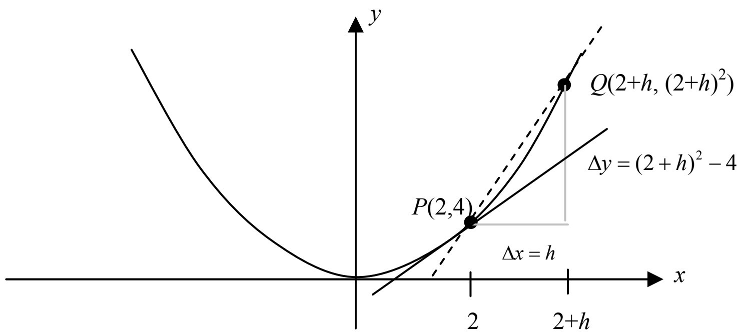
$$m = \frac{250 - 0}{32 - 9} = \frac{250}{23} = 10.87 \text{ flies/day}$$

As *Q*→ *P,* the average growth rates for increasingly smaller time intervals approach the slope of the tangent to the curve at *P*. Therefore the slope of tangent line is the rate of change on day 23.

Def: The *rate of change of y = f(x)* at $x = a$ is the slope of the tangent line to the curve at $x = a$.

So how do you find a tangent to a curve $y = f(x)$?

Ex. Find the slope of the parabola $y = x^2$



$$\text{Secant Slope } = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{h}$$

$$= \frac{h^2 + 4h + \cancel{4} - \cancel{4}}{h}$$

Everything without an *h* will cancel in the top (ALWAYS!!!)

$$= \frac{h^2 + 4h}{h} = h + 4$$

$$\therefore \lim_{Q \to P} (\text{secant slope}) = \lim_{h \to 0} (h + 4) = 4$$

Using the Point-Slope formula:

$y - y_1 = m(x - x_1)$

$y - 4 = 4(x - 2)$

$y - 4 = 4x - 8$

$y = 4x - 4$

- So, to find the tangent to a curve $y = f(x)$ at a point $(a, f(a))$, we use a similar procedure.

<u>Def</u>: *Slope of a Curve at a Point*

The <u>*slope of the curve*</u> $y = f(x)$ at a point $(a, f(a))$ is the number

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists

- The ***tangent to the curve*** at P is the line through P with this slope

Ex. Let $f(x)=\frac{1}{x}$

(a) Find the slope of the curve at $x = a$.

(b) Where does the slope = $-\frac{1}{4}$?

(c) What happens to the tangent to the curve at $(a, \frac{1}{a})$ for different values of a?

Solution:

(a) $m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\bullet\frac{x(x+h)}{x(x+h)}=\lim_{h\to 0}\frac{x-(x+h)}{xh(x+h)}=\lim_{h\to 0}\frac{-h}{xh(x+h)}=\lim_{h\to 0}\frac{-1}{x(x+h)}=-\frac{1}{x^2}$

at $x=a$: $m=-\frac{1}{a^2}$

Everything without an h will cancel in the top (ALWAYS!!!)

(b)

$-\frac{1}{a^2}=-\frac{1}{4}$

$4=a^2$

$a=\pm 2$

(c)

As $a\to\pm\infty$, the slope approaches 0

As $a\to 0^-$, the slope approaches $-\infty$

As $a\to 0^+$, the slope approaches $+\infty$

Def: The ***normal line*** to a curve at a point is the line perpendicular to the line tangent to the curve.

Ex. Write an equation of the normal to the curve $f(x)=4-x^2$ at $x = 1$

$$m=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{4-(x+h)^2-(4-x^2)}{h}$$

$$=\lim_{h\to 0}\frac{4-(x^2+2xh+h^2)-4+x^2}{h}$$

$$=\lim_{h\to 0}\frac{4-x^2-2xh-h^2-4+x^2}{h}$$

Everything without an h will cancel in the top

$$=\lim_{h\to 0}\frac{h(-2x-h)}{h}$$

$$=\lim_{h\to 0}(-2x-h)=-2x$$

At $x = a$:

$m=-2(1)=-2$

Slope of Normal: $m_\perp=\frac{1}{2}$

Using Point-Slope at (1, 3):

$y-3=\frac{1}{2}(x-1)$

Def: **Speed**

Let $y = f(t)$ be the position function of an object in motion. Its *instantaneous speed* at any time t is the ***instantaneous rate of change*** of position with respect to time at t.

$$\text{Speed}=\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

Ex. Find the speed of the rock falling if the position function is $f(x)=16t^2$ at time $t = 1$.

$$Speed=\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

$$=\lim_{h\to 0}\frac{16(t+h)^2-16t^2}{h}$$

$$=\lim_{h\to 0}\frac{16(t^2+2th+h^2)-16t^2}{h}$$

Everything without an h will cancel in the top

$$=\lim_{h\to 0}\frac{16t^2+32th+16h^2-16t^2}{h}$$

$$=\lim_{h\to 0}\frac{h(32t+16h)}{h}$$

$$=\lim_{h\to 0}(32t+16h)=32t$$

At $t = 1$: Speed = 32(1) = 32

Homework$_1$: Pg. 87–89 #1–6, 9–12, 15, 19, 20, 23

Homework$_2$: Pg. 88–89 # 24–32