Chapter 3 – Derivatives

3.1 Derivative of a Function

Def: The Formal Definition of the Derivative

The *derivative* of the function $f(x)$ with respect to the variable x is the function $f'(x)$, pronounced "*f prime of x*", whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Note: From last chapter, this is the slope of the curve or the slope of the tangent to the curve.

- The domain of f', the set of points in the domain of f for which the limit exists, may be smaller than the domain of f.
- If $f'(x)$ exists, we say that f has a derivative (is differentiable) at x.
  - If a function is differentiable at every point of its domain, then it is called a differentiable function.

Ex. Differentiate (find the derivative of)

1. $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(3x^2+3xh+h^2)}{\cancel{h}}$$
$$\boxed{f'(x) = 3x^2}$$

2. $f(x) = x^2 - 4x$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - 4(x+h) - (x^2-4x)}{h}$$
$$f'(x) = \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{4x}-4h-\cancel{x^2}+\cancel{4x}}{h}$$
$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h-4)}{\cancel{h}}$$
$$\boxed{f'(x) = 2x-4}$$

3. $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \longrightarrow f'(x) = \frac{1}{\sqrt{x}+\sqrt{x}}$$
$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

4. $f(x) = \dfrac{4}{x}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{4}{x+h} - \frac{4}{x}}{h} \cdot \frac{x(x+h)}{x(x+h)}$$
$$= \lim_{h\to 0} \frac{4x-4(x+h)}{hx(x+h)}$$
$$= \lim_{h\to 0} \frac{\cancel{4x}-\cancel{4x}-4\cancel{h}}{\cancel{h}x(x+h)}$$
$$= \lim_{h\to 0} \frac{-4}{x(x+h)} \longrightarrow \boxed{f'(x) = \frac{-4}{x^2}}$$

Def: Derivative at a Point (Alternate Definition)

The *derivative* of the function $f(x)$ at the point $x = a$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

Ex. Differentiate $f(x) = x^2$ at x = 5 using the alternate definition

$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x \to 5} \frac{x^2 - 25}{x - 5} = \lim_{x \to 5} \frac{(x+5)\cancel{(x-5)}}{\cancel{x-5}}$$

$$\boxed{f'(5) = 10}$$

If stopping here:
HW: p.105-106 #1-6, 11, 12, 15

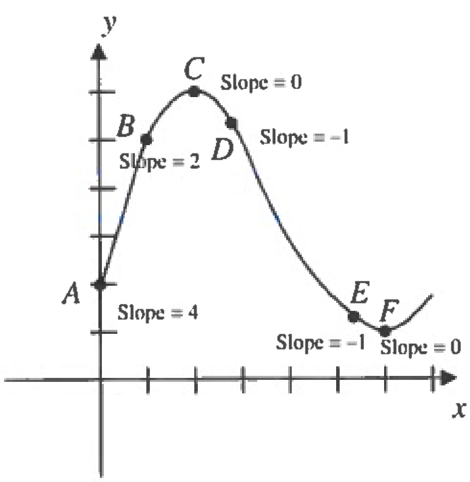
Notation: These are all $f'(x)$

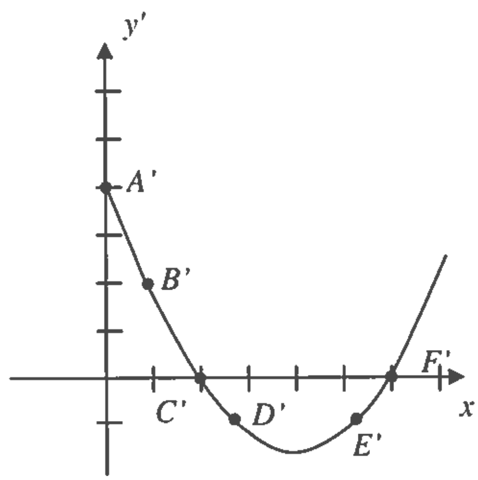
There are many ways to denote the derivative of a function $y = f(x)$. Other common notations are

y'	"y prime"	Handy – but does not name the independent variable (x)
$\frac{dy}{dx}$	"dy dx" "the derivative of y with respect to x"	Names both variables and uses d for derivative
$\frac{df}{dx}$	"df dx" "the derivative of f with respect to x"	Emphasizes the function's name
$\frac{d}{dx}f(x)$	"d dx of f at x" "the derivative of f at x"	Emphasizes the idea that differentiation is an operation performed on f

Graphing f' from f:

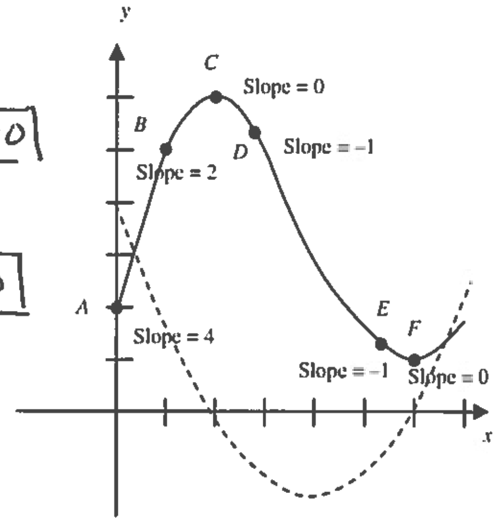
- Suppose we wanted to graph the function $y = f'(x)$ based on the graph of $y = f(x)$ to below left.
- Basically, plot a new point $(a, f'(a))$ on another set of axes. This will be the graph of $y = f'(x)$.
    - After plotting each point (the x coordinate with its slope of tangent line (y'), we draw a smooth curve through the points.





- If we superimpose the derivative function onto the original graph, we get the graph below. (f' is the dashed line).

- Notice the different "sections" of the two curves:



  - From A to C, $f(x)$ is increasing. What do you notice about $f'(x)$ on the same domain values of x?

    $f'(x)$ is above x-axis $\longrightarrow$ $\boxed{f'(x)>0}$

  - From C to F, $f(x)$ is decreasing. What do you notice about $f'(x)$ on the same domain values of x?

    $f'(x)$ is below x-axis $\longrightarrow$ $\boxed{f'(x)<0}$

  - What can you say about the slope of an increasing function? A decreasing function?

    $\boxed{\begin{array}{l}\text{inc}: f'(x)>0\\ \text{dec}: f'(x)<0\end{array}}$

  - What is the slope of the tangent at C and F? What do you notice happens with $y = f(x)$ around these two points? slope $= 0$ $\boxed{f(x) \text{ turns}}$

<u>One-Sided Derivatives</u>:

- A function $y = f(x)$ is <u>differentiable on a closed interval [a,b]</u> if it has a derivative at every interior point on the interval and if the limits:

$$\lim_{h\to0^+}\frac{f(a+h)-f(a)}{h} \text{ [the right-hand derivative at } a\text{] or } \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

$$\lim_{h\to0^-}\frac{f(b+h)-f(b)}{h} \text{ [the left-hand derivative at } b\text{] or } \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$$

exist at the endpoints

ex. Show the following function has left-hand and right-hand derivatives at $x = 0$, but no derivative there.

$$f(x)=\begin{cases}x^2, x\le 0\\ 2x, x>0\end{cases}$$

use alternate def

<u>left</u>: $\lim_{x\to0^-}\frac{f(x)-f(a)}{x-a} = \lim_{x\to0^-}\frac{x^2-0^2}{x-0} = \lim_{x\to0^-}\frac{x^{\cancel{2}}}{\cancel{x}} = \lim_{x\to0^-} x = 0$

$\neq$

<u>right</u>: $\lim_{x\to0^+}\frac{2x-2(0)}{x-0^+} = \lim_{x\to0^+}\frac{2\cancel{x}}{\cancel{x}} = \lim_{x\to0^+} 2 = 2$

$\therefore \boxed{f'(x) \text{ DNE}}$

Homework: pg. 105-106 Ex. #1-9odd, 10-18e, 25, 26, 31
Day 2 (if needed): 2-8e, 11-19o, 20, 27, 32 ← do if stopped early (Day 2)

3.2 Differentiability

Ex. Find the derivative for each of the following:

1. $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}\ 1}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}} \rightarrow \boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

2. $f(t) = \dfrac{2}{t}$

$$f'(t) = \lim_{h \to 0} \frac{\frac{2}{t+h} - \frac{2}{t}}{h} \cdot \frac{t(t+h)}{t(t+h)}$$

$$= \lim_{h \to 0} \frac{2t - 2(t+h)}{ht(t+h)}$$

$$= \lim_{h \to 0} \frac{\cancel{2t} - \cancel{2t} - 2\cancel{h}}{\cancel{h}t(t+h)}$$

$$= \lim_{h \to 0} \frac{-2}{t(t+h)} = \frac{-2}{t \cdot t}$$

$$\boxed{f'(t) = -\frac{2}{t^2}}$$

3. $f(x) = 2x^2 + x - 1$

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)^2 + (x+h) - 1 - (2x^2 + x - 1)}{h}$$

$$= \lim_{h \to 0} \frac{2(x^2 + 2xh + h^2) + x + h - 1 - 2x^2 - x + 1}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{+x} + h \cancel{-1} \cancel{-2x^2} \cancel{-x} \cancel{+1}}{h}$$

$$= \lim_{h \to 0} \frac{h(4x + 2h + 1)}{h}$$

$$= \lim_{h \to 0} 4x + \overset{\to 0}{2h} + 1$$

$$\boxed{f'(x) = 4x + 1}$$

4. $f(x) = x + \dfrac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{x + h + \frac{1}{x+h} - \left(x + \frac{1}{x}\right)}{h} \cdot \frac{x(x+h)}{x(x+h)}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{x^2(x+h)} + h\,x(x+h) + x - \cancel{x^2(x+h)} - (x+h)}{hx(x+h)}$$

$$= \lim_{h \to 0} \frac{hx(x+h) \cancel{+x} \cancel{-x} - h}{hx(x+h)}$$

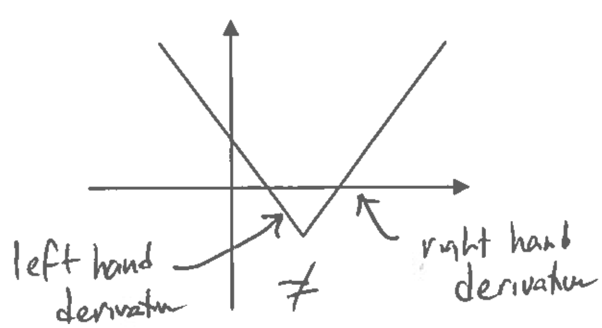
$$= \lim_{h \to 0} \frac{\cancel{h}[x(x+h) - 1]}{\cancel{h}x(x+h)}$$

$$= \lim_{h \to 0} \frac{x^2 + \overset{\to 0}{xh} - 1}{x(x+\cancel{h})} = \frac{x^2 - 1}{x \cdot x}$$

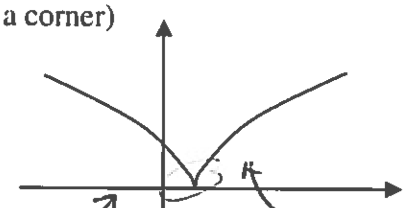
$$\boxed{f'(x) = \frac{x^2 - 1}{x^2}}$$

Property: $y = f(x)$ will have no derivative at $x = a$ where the graph has:

1. A corner: One-sided derivatives differ



2. a cusp: where the slopes of the secant lines approach ∞ from one side and –∞ from the other side (an extreme case of a corner)

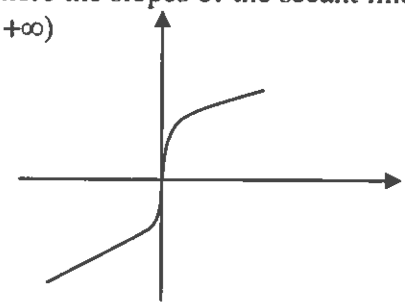


slope approaches vertical but negative $m \to -\infty$

slope approaches vertical but positive $m \to +\infty$

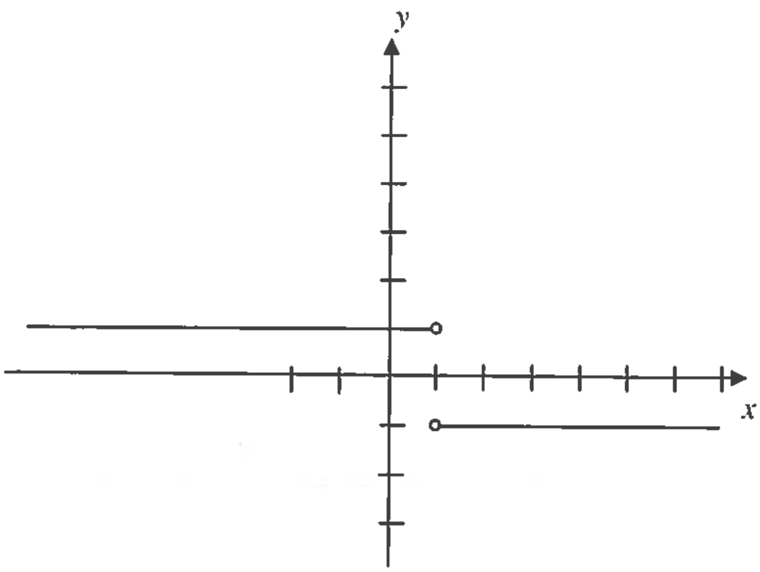
$\neq$

3. a vertical tangent: where the slopes of the secant lines approach either +∞ or –∞ from both sides (this example: +∞)



$m \to \pm\infty$

4. A point of discontinuity:



To find where a function is not differentiable:

- If it has an absolute value, then set the inside of the absolute value equal to 0 and solve for x

  ex $f(x) = |x-4|$
  Disc @ $x-4=0$
  $x=4$

  $f(x) = |x^2-4|$
  disc when $x^2-4=0 \to x=\pm 2$

- Most functions in calculus will be differentiable, which means *no corners, no cusps,no points of discontinuity*, or *vertical tangent lines* within their domains. Their curves will be smooth with a well-defined slope at each point.

Differentiability Implies Continuity

Theorem: If f has a derivative at $x = a$, then f is continuous at $x = a$.

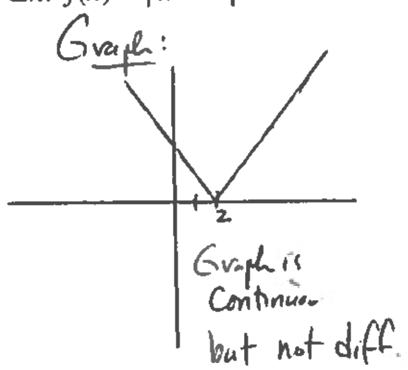
Proof: We will show that $\lim_{x\to a} f(x) = f(a)$ or better yet: $\lim_{x\to a}[f(x) - f(a)] = 0$

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a}\left[(f(x)-f(a))\cdot\frac{x-a}{x-a}\right]$$
$$= \lim_{x\to a}\left[(x-a)\cdot\frac{f(x)-f(a)}{x-a}\right]$$
$$= \lim_{x\to a}(x-a)\cdot\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$
$$= 0\cdot f'(a)$$
$$= 0$$

This is by definition, $f'(a)$

Although Differentiability implies Continuity, the converse (Continuity implies Differentiability) is not true.

Ex. $f(x) = |x - 2|$



So $D \to C$ ("implies")
but
$C \not\to D$

To summarize the relationship between Differentiability and Continuity:

1. If a function is differentiable at $x = a$, then it is continuous at $x = a$ $\quad D \to C$
2. If a function is continuous at $x = a$, then it does not have to be differentiable at $x = a$ $\quad C \not\to D$
3. If a function is NOT continuous at $x = a$, then it does NOT have a derivative at $x = a$.

$\sim C \to \sim D$

Derivatives with Calculators

You can find the derivative of a function at a point using your TI-83+ using the "nderiv" function. The command is:

nderiv($f(x)$, x, a)

ex. nderiv(x^2,x, 4) → 8
nderiv(sin x, x, $\pi/4$) → 0.7071

You are only to use this to check your answers or if the questions instructs you to use it (NDER in book)

Homework: Pg. 114 #1–10, 17

3.3 Rules for Differentiation (a.k.a THE SHORTCUT!!!)

Rule #1: Derivative of a Constant Function

If f is the function with the constant value c, $f(x) = c$, then

$$\frac{df}{dx} = f'(x) = 0$$

Rule #2: Power Rule for Positive Integer Power of x

If $f(x) = x^n$ and n is a positive integer, then

$$\frac{df}{dx} = f'(x) = nx^{n-1}$$

To differentiate x^n, multiply the exponent n by x to 1 less than the n.

Ex. Find the derivative of

1. $f(x) = x^5$
   $f'(x) = 5x^4$
2. $f(x) = x^2$
   $f'(x) = 2x$
3. $f(x) = x^{12}$
   $f'(x) = 12x^{11}$

Rule #3: The Constant Multiple Rule

If f is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

Ex
$f(x) = 4x^3$
$f'(x) = 4 \cdot 3x^2$
$\boxed{f'(x) = 12x^2}$

Rule #4: The Sum and Difference Rule

If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

Ex. Find the derivative of

1. $f(x) = x^3 - 4x^2 + 5x - 6$
   $f'(x) = 3x^2 - 8x + 5$
2. $f(x) = 8x^5 - 4x^4 - 9x^3 - 12x^2 - 11x + 10$
   $f'(x) = 40x^4 - 16x^3 - 27x^2 - 24x - 11$

Rule #5: The Product Rule

The product of two differentiable functions u and v is differentiable, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \text{Shorthand: } uv' + vu'$$

Alternate Writing: If $f(x) = u(x)\cdot v(x)$, then $f'(x) = u(x)v'(x) + v(x)u'(x)$

ex. Find the derivative of

1. $f(x) = (x^2+1)(x^3+3)$

$u \quad v$

$u\,v' + v\,u'$

$f'(x) = (x^2+1)\cdot 3x^2 + (x^3+3)\cdot 2x$

$f'(x) = 3x^4 + 3x^2 + 2x^4 + 6x$

$\boxed{f'(x) = 5x^4 + 3x^2 + 6x}$

2. $f(x) = (x^3+3x)(2x^2+3x+3)$

$u \quad v$

$\boxed{f'(x) = (x^3+3x)(4x+3) + (2x^2+3x+3)(3x^2+3)}$

Rule #6: The Quotient Rule

At a point where $v \neq 0$, the quotient $y = \frac{u}{v}$ of two differentiable function is differentiable, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \qquad \text{Shorthand: } \frac{vu'-uv'}{v^2}$$

ex. Find the derivative of $f(x) = \frac{x^2-1}{x^2+1}$ $\begin{matrix} u \\ v \end{matrix}$

$\frac{vu'-uv'}{v^2}$

$f'(x) = \frac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$

$f'(x) = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2}$

$\boxed{f'(x) = \frac{4x}{(x^2+1)^2}}$

Ex. Write the equation of the line tangent to $f(x) = \frac{x}{x-1}$ at $x = 2$

$y = \frac{2}{2-1} = 2$ Pt. $(2,2)$

$\frac{vu'-uv'}{v^2}$

$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2}$

$f'(x) = \frac{x-1-x}{(x-1)^2}$

$f'(x) = \frac{-1}{(x-1)^2}$

@ $x=2$: $f'(x) = \frac{-1}{(2-1)^2} = \frac{-1}{1} = -1$ ← slope

$y - y_1 = m(x - x_1)$

$\boxed{y - 2 = -1(x-2)}$

Rule #7: Power Rule for Negative Integer Power of x

If $f(x) = x^n$ and n is a negative integer and $x \neq 0$, then

$$\frac{df}{dx} = f'(x) = nx^{n-1}$$

ex. Find the derivative of

1. $y = x^{-6}$

$\boxed{y' = -6x^{-7}}$

$\boxed{y' = \frac{-6}{x^7}}$

2. $y = 5x^{-3}$

$\boxed{y' = -15x^{-4}}$

3. $y = \frac{6}{x^5} = 6x^{-5}$

$\boxed{y' = -30x^{-6}}$

Higher Order Derivatives

- The derivative $y' = \frac{dy}{dx}$ is called the *first derivative* of y with respect to x. If the first derivative is also a differentiable function, then

$$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$ is called the *second derivative of y with respect to x*.

y'' is pronounced "y double–prime"

- y''' is the third derivative of y with respect to x

$$y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

- Continuing, we end up with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)}$$ The n^{th} *derivative of y with respect to x* $\left(\frac{d^n y}{dx^n}\right)$

Ex. Find the first 4 derivatives for $f(x) = x^4 + 2x^3 - 5x^2 + 7x - 10$

$f'(x) = 4x^3 + 6x^2 - 10x + 7$

$f''(x) = 12x^2 + 12x - 10$

$f'''(x) = 24x + 12$

$f^{IV}(x) = 24$

ex. If $f(x) = \frac{3x^2-4x+6}{x}$, find $f''(x)$

Could do quotient rule TWICE!

$f(x) = 3x - 4 + 6x^{-1}$

$f'(x) = 3 - 6x^{-2}$

$f''(x) = 12x^{-3}$

$\boxed{f''(x) = \frac{12}{x^3}}$

ex. Find tangents to $y = x^3 + x$ when slope $= 4$. What is smallest slope of curve? When does it occur?

$y' = 3x^2 + 1 = 4$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

@ $x = 1$: $y = 1^3 + 1 = 2$

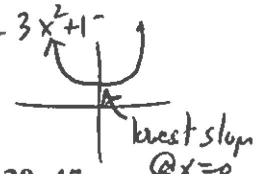
$y' = 4$

$\boxed{y - 2 = 4(x-1)}$

@ $x = -1$: $y = (-1)^3 + (-1) = -2$

$\boxed{y + 2 = 4(x+1)}$

$y' = 3x^2 + 1$



$\therefore$ smallest slope:

$y' = 3(0)^2 + 1$

$y' = 1$

Homework: Day 1: Pg. 124–125 #1–19o, 22, 23, 25, 31, 33

Day 2: Pg. 124–125 #4, 10, 12, 18, 20, 24, 27, 28, 34, 36, 39, 45

Quiz: Product/Quotient Rules

Worksheet 3.1 – 3.3

Quiz: Derivatives I

3.4 Velocity and Other Rates of Change

Def: *Instantaneous Rates of Change*

The *Instantaneous Rate of Change* of f with respect to x at a is the derivative.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Ex. Find the rate of change of the area A of a circle with respect to its radius r. What is the rate of change of A when $r = 3$, $r = 6$, and when $r = 8$?

$A(r) = \pi r^2$

$A'(r) = 2\pi r$

@ $r=3$: $A'(3) = 6\pi$

$r=6$: $A'(6) = 12\pi$

$r=8$: $A'(8) = 16\pi$

Motion along a Line

Def: Suppose that an object is moving along a coordinate line (the x-axis) so that we know its position, called s, on that line as a function of time t.

$$s = f(t)$$

The *displacement* of the object over the time interval t to $t + \Delta t$ is

$$\Delta s = f(t+\Delta t) - f(t)$$

Def: The *average velocity* of an object over that time interval is

$$v_{avg} = \frac{\text{displacement}}{\text{time}} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Def: The *instantaneous velocity* is the velocity of the object at the exact instant t.

This can be found by taking the limit of the average velocity as $\Delta t \to 0$.

Therefore, the velocity of an object at time t is the derivative of the position function $s = f(t)$ with respect to time. At time t, the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = s'(t)$$

$$\boxed{v(t) = s'(t)}$$

ex. A particle is moving along a line such that its position is given by the function $s(t) = 4t^3 + 3t^2 - 12t$. What is the velocity of the particle at $t = 1, 2, 5$, and 6 seconds.

$V(t) = 12t^2 + 6t - 12$

$V(1) = 12 + 6 - 12 = 6$

$V(2) = 12 \cdot 4 + 6 \cdot 2 - 12 = 48$

$V(5) = 12 \cdot 25 + 6 \cdot 5 - 12 = 318$

$V(6) = 12 \cdot 36 + 6 \cdot 6 - 12 = 456$

Def: Speed is the absolute value of velocity.

$$Speed = |\, v(t)\, |$$

The difference between speed and velocity is that velocity gives you two values: how fast the object is moving and in which direction. Speed does not.

Def: *Acceleration* measures how quickly an object picks up or loses speed. Acceleration is the derivative of velocity with respect to time. If a body's velocity at time t is $v(t) = \frac{ds}{dt}$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

$a(t) = v'(t) = s''(t)$

Note: Acceleration is the position function's 2nd Derivative.

Ex. Using the previous example, what is the acceleration of the particle at $t = 1, 2, 5$, and 6?

$a(t) = 24t + 6$

$a(1) = 24 + 6 = 30$

$a(2) = 48 + 6 = 54$

$a(5) = 120 + 6 = 126$

$a(6) = 144 + 6 = 150$

Some Constants to Know About the Earth:

Experimental and theoretical investigations revealed that the distance a body released from rest ($v = 0$) falls freely is proportional to the square of the amount of time it has fallen.

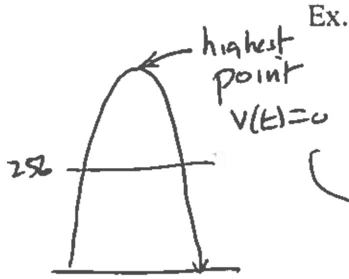
$s = \frac{1}{2}gt^2$, where s is distance, g is acceleration due to gravity, and t is time. Units are determined but units used to measure s and t

Free–Fall Constants:

English Units: $g = 32\frac{ft}{sec^2}$, $s = \frac{1}{2}(32)t^2 = 16t^2$ (feet)

Metric Units: $g = 9.8\frac{m}{sec^2}$, $s = \frac{1}{2}(9.8)t^2 = 4.9t^2$ (meters)

Units for acceleration would be ft/sec^2 is read "feet per second squared"



Ex. A dynamite blast propels a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph). It reaches a height of $s = 160t - 16t^2$ feet after t seconds.

(a) How high does the rock go?
(b) What is the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?
(c) What is acceleration of the rock at any time t during its flight (after the blast)?
(d) When does the rock hit the ground? How fast is it traveling when it hits the ground?

(a) $V(t) = 160 - 32t = 0$

$\frac{-32t}{-32} = \frac{-160}{-32}$

$t = 5$ ← when

@ $t=5$: $s(5) = 160(5) - 16(5)^2 = \boxed{400 \text{ ft}}$

(b) $s = 160t - 16t^2 = 256$

$0 = 16t^2 - 160t + 256$

$0 = 16(t^2 - 10t + 16)$

$0 = 16(t-2)(t-8)$

$t=2, \quad t=8$

$V(2) = 160 - 32(2)$ | $V(8) = 160 - 32(8)$

$\boxed{V(2) = 96}$ | $\boxed{V(8) = -96}$ ← coming down

(c) $a(t) = -32$ ∴ always -32

(d) Rock on ground when $s(t) = 0$

$160t - 16t^2 = 0$

$16t(10-t) = 0$

$16t = 0$ | $10 - t = 0$

$t = 0$ | $t = 10$

start | hits ground

$V(10) = 160 - 32(10)$

$\boxed{V(10) = -160 \text{ ft/sec}}$

Homework: Pg. 135-137 #1-6, 9, 12, 13, 19, 20, 22, 24, 36

Worksheet: Pos, Vel., Accel. | Worksheet 3.1-3.4B

3.4b Marginal Cost and Revenue (Economics)

- Engineers use velocity and acceleration to refer to the derivatives of functions describing motion.
- Economists use derivatives as well for rates of change. This is referred to as *marginals*.
- In manufacturing:
  - The cost of production $c(x)$ is a function of x, the number of units produced.
  - The *marginal cost of production* is the rate of change of the cost with respect to the level of production.
    - In other words: $\frac{dc}{dx}$ or $c'(x)$
  - Revenue is the amount of money made from the selling of a product. It is a function of the number of units sold, x. $r(x)$
  - *Marginal Revenue* is the rate of change of the revenue with respect to the level of production, or $r'(x)$

Ex. Suppose the cost to produce x radiators when 8 to 10 radiators are produced is given by $c(x) = x^3 - 6x^2 + 15x$ and the dollars of revenue from selling these x radiators is $r(x) = x^3 - 3x^2 + 12x$. If the shop produces 10 radiators per day, find the marginal cost and marginal revenue.

Marginal Cost: $c'(x) = 3x^2 - 12x + 15$

$c'(10) = 300 - 120 + 15 = \boxed{\$195}$

Marginal Revenue: $r'(x) = 3x^2 - 6x + 12$

$r'(10) = 300 - 60 + 12 = \boxed{\$252}$

Def: Profit - amount of money made based on revenue and cost

$p(x) = r(x) - c(x)$ → if $p(x) > 0$ → profit (making \$)

$p(x) < 0$ - loss

$p(x) = 0$ - Breaking even

∴ Marginal Profit: $p'(x) = r'(x) - c'(x)$

Ex. The total cost, in dollars, of producing x food processors is $C(x) = 2000 + 50x - 0.5x^2$.

(a) What is the average cost for each food processor if you produce 21 food processors?

(b) Find the exact cost of producing the 21st food processor.

(c) Use the marginal cost to approximate the cost of producing the 21st food processor.

(a) Avg cost $= \frac{\text{total cost}}{\#\text{of items}} = \frac{C(21)}{21} = \frac{2829.50}{21} = \boxed{\$134.74}$

(b) Total cost of 20: $C(20) = \$2800$

Total cost of 21: $C(21) = \$2829.50$

∴ cost of 21st $= C(21) - C(20)$

$= \boxed{\$29.50}$

(c) marginal cost $= c'(x) = 50 - x$

$c'(21) = 50 - 21 = \$29$

estimate

Homework: Pg. 136-138 #18, 27-29, 31, 32, 34, 36, 37, 38, 42-45, 47

Quiz: 3.1-3.4 Derivative II

Change: p. 136-138 # 8, 10, 14, 16, 18, 23, 27, 28, 31, 37, 42-45

3.5 Derivatives of Trig Functions (Short Version)

Th: The Derivative of the 6 Trig Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

Ex. Find the derivative of each of the following.

1. $y = x^2 \sin x$ ← Product Rule!! $uv'+vu'$

$y' = x^2\cos x + \sin x\cdot(2x)$

$\boxed{y' = x^2\cos x + 2x\sin x}$

2. $f(x) = \dfrac{\cos x}{1-\sin x}$ $\begin{matrix} u \\ v\end{matrix}$ Quotient Rule $\dfrac{vu'-uv'}{v^2}$

$f'(x) = \dfrac{(1-\sin x)(-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$

$f'(x) = \dfrac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} = \dfrac{-\sin x + 1}{(1-\sin x)^2}$ ← Same ←

$\boxed{f'(x) = \dfrac{1}{1-\sin x}}$

3. $f(x) = \dfrac{\tan x}{x}$

$f'(x) = \dfrac{x\sec^2 x - \tan x}{x^2}$

4. $f(x) = \sin x \tan x$ $uv'+vu'$

$f'(x) = \sin x(\sec^2 x) + \tan x \cos x$

$f'(x) = \sin x \sec^2 x + \sin x$

Ex. Find y'' if $y = \sec x$

$y' = \underset{u}{\sec x}\,\underset{v}{\tan x}$

$uv'+vu'$

$y'' = \sec x \cdot \sec^2 x + \tan x \sec x \tan x$

$\boxed{y'' = \sec^3 x + \sec x \tan^2 x}$

$\sec^2 x = 1 + \tan^2 x$

$\sec^2 x - 1 = \tan^2 x$

$y'' = \sec^3 x + \sec x(\sec^2 x - 1)$

$y'' = \sec^3 x + \sec^3 x - \sec x$

$\boxed{y'' = 2\sec^3 x - \sec x}$

Ex. Find y' if $y = \sin 2x$ (hint use formula for $y = \sin 2x$)

$y = 2\sin x \cos x$ $\qquad uv' + vu'$

$y' = 2\left[\sin x(-\sin x) + \cos x(\cos x)\right]$

$y' = 2\left[-\sin^2 x + \cos^2 x\right]$

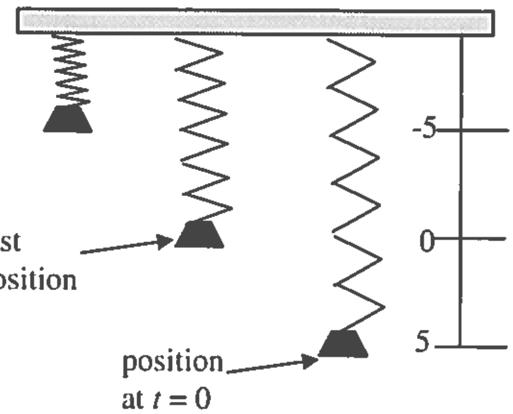
$y' = 2\left[\cos^2 x - \sin^2 x\right]$ ← this is ok → $\boxed{y' = 2\cos 2x}$

Def: The motion of a weight bobbing up and down at the end of a spring is an example of *simple harmonic motion*.

ex. A weight is hanging from a spring (see diagram right) is stretched 5 units beyond its rest point (s=0) and released at time t = 0 to bob up and down. Its position at any later time t is

$$s = 5\cos t$$

What is the velocity and acceleration at time t?



$v(t) = -5\sin t$

$a(t) = -5\cos t$

note: $a(t) = -s(t)$ – the acceleration is equal to the opposite of the position

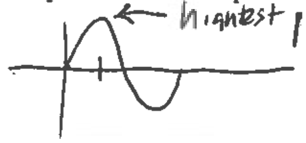
Def: Jerk is the derivative of acceleration. If a body's position at time t is $s(t)$, the body's jerk at time t is:

$$j(t) = \frac{da}{dt} = \frac{d^3 s}{dt^3} = s'''(t)$$

$j(t) = a'(t)$

ex. What is the jerk force on the weight in the previous example at time t? When is the jerk force at its highest?

$j(t) = 5\sin t$



← highest point @ $\boxed{t = \frac{\pi}{2}}$

Homework: Day 1: Pg 146 #1–23o, 27, 29
Day 2: Pg 146 #2-26e, 30

$\boxed{\text{Quiz: 3.1–3.5}}$ (Don't if you did 3.1–3.4)

Quiz: Derivatives III

3.6 Chain Rule

recall: A composite function is defined as a function containing another function.

$$f(g(x)) = f[g(x)] = (f \circ g)(x)$$

Suppose you have the function $y = 2(3x - 5)$ and you were asked to find the derivative.

The function $y = 2(3x - 5)$ is a composite of the functions $y=2u$ and $u = 3x - 5$

Distributing the 2 gets $y = 6x - 10$

Suppose we take the derivative of all three equations:

$y = 6x - 10$	$y=2u$	$u = 3x - 5$
$\frac{dy}{dx} = 6$	$\frac{dy}{du} = 2$	$\frac{du}{dx} = 3$

See a relationship between the numbers? $6 = 2 \times 3$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Does this always work??

Ex. Find the derivative of $y = (3x^2 + 1)^2$

$y = u^2 \quad u = 3x^2 + 1$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{dy}{dx} = 2u \cdot 6x$

$\frac{dy}{dx} = 12ux$

$= 12(3x^2+1)x$

$\frac{dy}{dx} = 36x^3 + 12x$

$y = 9x^4 + 6x^2 + 1$

$y' = 36x^3 + 12x$

yes

Rule #8: The Chain Rule

If f is differentiable at the point $u = g(x)$ and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then : $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Ex. An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$.
Find the velocity of the object as a function of t.

$x = \cos u \qquad u = t^2 + 1$

$$V: \frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$$

$$v(t) = -\sin u \cdot 2t$$

$$\boxed{v(t) = -2t\sin(t^2+1)}$$

you do it using the idea of working inward

$v(t) = -\sin(t^2+1) \cdot 2t$

take the derivative of the outside function, leave inside alone.

then multiply by derivative of inside

Keep going

Ex. Find the derivative of $f(t) = \tan(5 - \sin 2t)$

$$f'(x) = \sec^2(5-\sin 2t) \cdot (-\cos 2t) \cdot 2$$

$$\boxed{f'(x) = -2\sec^2(5-\sin 2t)\cos 2t}$$

Power Chain Rule: If $y = [f(x)]^n$, then $y' = n[f(x)]^{n-1} f'(x)$

Power Chain Rule (alternate form): $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

One more look at it

If $f(x) = u^n$, where u is a differentiable function of x, then :

$$f'(x) = nu^{n-1} \cdot u'$$

Ex. Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \frac{\pi}{3}$.

$y = (\sin x)^5$

$$y' = 5(\sin x)^4 \cos x$$

$$y' = 5\sin^4 x \cos x$$

$$@\ x = \pi/3: \ y' = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = 5\left(\frac{9}{16}\cdot\frac{1}{2}\right) = \frac{45}{32}$$

Ex. Show the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

$$y = (1-2x)^{-3}$$

$$y' = -3(1-2x)^{-4}(-2)$$

$$y' = +6(1-2x)^{-4}$$

$$y' = \frac{6}{(1-2x)^4}$$

top is always positive

for all $x \neq 1/2$, $(1-2x)^4 > 0$

$\therefore$ y' is always positive

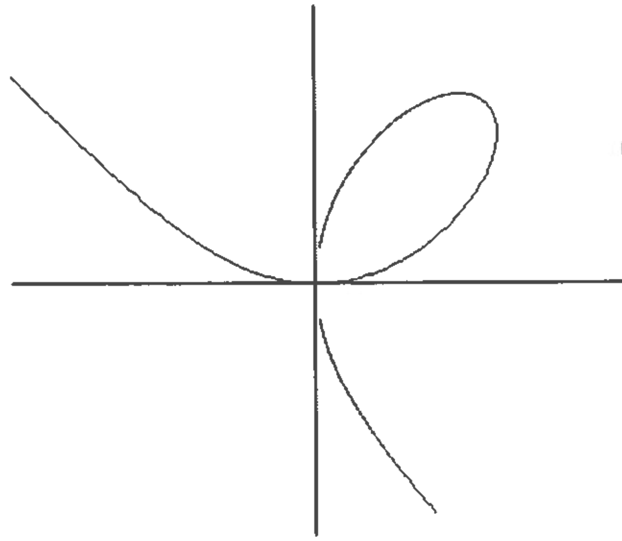
Homework: Pg. 153 #1-19o, 26, 31

Day 2: p.153-156 # 4-28(4's), 33-39o, 49, 53-56, 59, 72, 73

$\boxed{\text{Quiz: 3.1-3.6}}$

3.7 Implicit Differentiation

To the right is a curve called a *folium*



- First discussed by Descartes in 1638
- The equation of the curve is:

  $$x^3 + y^3 - 9xy = 0$$

- But this equation can not be solved for y so finding the slope of the tangent at any point can not be done the way we have before.
- In previous sections, we always had an equation of the form $y = f(x)$
  - This kind of differentiation is called *explicit differentiation.*

Consider the equation $xy = 1$

To find $\frac{dy}{dx}$, we would rewrite the above equation as

$$y = \frac{1}{x} \text{ or}$$

$$y = x^{-1}$$

Finding the derivative: $\frac{dy}{dx} = -1x^{-2} = \frac{-1}{x^2}$

There is another way to do this. This is called *implicit differentiation.*

We can differentiate (take the derivative) of both sides of $xy = 1$ before solving for y.

Product Rule $uv' + vu'$

$$\frac{d}{dx}(xy) = \frac{d}{dx}(1)$$

$$x\frac{dy}{dx} + y \cdot 1 = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

Substituting $y = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{-1/x}{x} \cdot \frac{x}{x}$$

complex fraction

$$\boxed{\frac{dy}{dx} = \frac{-1}{x^2}}$$

This agrees with the previous result using explicit differentiation.

- The advantage of implicit differentiation is that you don't have to solve the equation for y.
- So in general, you do not have to substitute back in for y. Leave the y in the equation unless otherwise stated.
- Use implicit differentiation if you can't solve for y.

Implicit Differentiation Process:

1. Differentiate both sides of the equation with respect to x
2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation
3. Factor out $\frac{dy}{dx}$
4. Solve for $\frac{dy}{dx}$

Ex. Find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$

$$5\cdot 2y\frac{dy}{dx} + \quad y\frac{dy}{dx} = 2x$$

$$10y\frac{dy}{dx} + \cos y\frac{dy}{dx} = 2x$$

$$\frac{\frac{dy}{dx}(10y + \cos y)}{10y+\cos y} = \frac{2x}{10y+\cos y}$$

$$\boxed{\frac{dy}{dx} = \frac{2x}{10y+\cos y}}$$

ex. Find the slope of the tangent line at (4,0) to the graph of $7y^4 + x^3y + x = 4$.

$$7y^4 + x^3y + x = 4$$

$$28y^3y' + (x^3y' + y\cdot 3x^2) + 1 = 0$$

$$28y^3y' + x^3y' + 3x^2y + 1 = 0$$

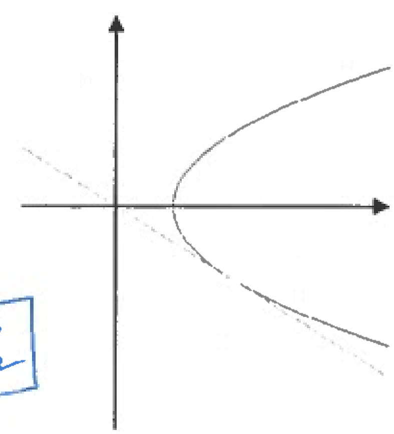
$$\frac{y'(28y^3 + x^3)}{28y^3+x^3} = \frac{-1-3x^2y}{28y^3+x^3}$$

$$y' = \frac{-1-3x^2y}{28y^3+x^3}$$

@ (4,0): $y' = \frac{-1-3(16)(0)}{28(0)^3+(4)^3}$

$$\boxed{y' = \frac{-1}{64}}$$

Ex. Find the slope of the tangent line at (2,–1) to $y^2 - x + 1 = 0$



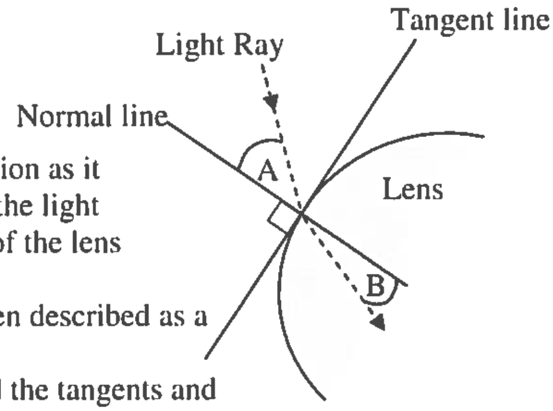
$$2yy' - 1 = 0$$

$$2yy' = 1$$

$$y' = \frac{1}{2y}$$

@ (2,-1): $y' = \frac{1}{2(-1)} = \boxed{-\frac{1}{2}}$

Lenses, Tangents, and Normal Lines



- In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens (the normal) at the point of entry.
- How a lens is curved (the profile of lenses) is often described as a quadratic curve.
  - We can use implicit differentiation to find the tangents and normals.

Ex. Find the normal and tangent to the ellipse $x^2 - xy + y^2 = 7$ at the point $(-1,2)$.

$$x^2 - xy + y^2 = 7$$

$$2x - (xy' + y) + 2yy' = 0$$

$$2x - xy' - y + 2yy' = 0$$

$$2yy' - xy' = y - 2x$$

$$\frac{y'(2y-x)}{2y-x} = \frac{y-2x}{2y-x}$$

$$y' = \frac{y-2x}{2y-x}$$

@(−1,2): $y' = \frac{2-2(-1)}{2(2)-(-1)} = \frac{2+2}{4+1} = \frac{4}{5}$

$$m_T = \frac{4}{5} \longrightarrow m_N = -\frac{5}{4}$$

$$\boxed{T:\ y-2 = \frac{4}{5}(x+1)}$$

$$\boxed{N:\ y-2 = -\frac{5}{4}(x+1)}$$

Derivatives of Higher Order

- Implicit differentiation can be used to find derivatives of higher order (2nd, 3rd, etc…)

Ex. Use implicit differentiation to find $\frac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$

$$4x^2 - 2y^2 = 9$$

$$8x - 4yy' = 0$$

$$-4yy' = -8x$$

$$y' = \frac{8x}{4y}$$

$$y' = \frac{2x}{y}$$

Quotient Rule! $\frac{vu' - uv'}{v^2}$

$$y'' = \frac{y \cdot 2 - 2xy'}{y^2}$$

$$y'' = \frac{2y - 2xy'}{y^2}$$

$$y'' = \frac{2y - 2x\left(\frac{2x}{y}\right)}{y^2} \cdot \frac{y}{y}$$

$$y'' = \frac{2y^2 - 4x^2}{y^3}$$

$$y'' = \frac{-9}{y^3}$$

why?

Rational Powers of Differentiable Functions (Fractional Exponents)

- We know the Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$ is true for any integer n (Rules 2 and 7).

Rule #9: Power Rule for Rational Powers of x

If n is any rational number, then:

$$\frac{d}{dx}x^n = nx^{n-1}$$

If $n < 1$, then the derivative does not exist at $x = 0$.

So this Rule works for ANY kind of exponent!

Ex. Find the derivative of each of the following:

1. $y = \sqrt{x} = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2}$

$y' = \frac{1}{2x^{1/2}}$

$\boxed{y' = \frac{1}{2\sqrt{x}}}$

2. $y = \sqrt[3]{x^2} = x^{2/3}$

$y' = \frac{2}{3}x^{-1/3}$

$y' = \frac{2}{3x^{1/3}}$

$\boxed{y' = \frac{2}{3\sqrt[3]{x}}}$

3. $y = (\cos x)^{-\frac{1}{5}}$

$y' = -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$

$y' = \frac{\sin x}{5(\cos x)^{6/5}}$

$y' = \frac{\tan x}{5\sqrt[5]{\cos x}}$ ← how?

Homework: Day 1: Pg. 162 #1-19odd, 23-35 odd, 37(a)
Day 2: Pg. 162 – 163 #2, 6, 10, 14, 16, 20, 24, 26, 28, 32, 36, 38(a), 41, 42, 43, 55

Day 3: pg. 162-164 #4, 8, 30, 44, 45(a), 48-57
Bonus (10pts) - #58

$\boxed{\text{Quiz: 3.1–3.7}}$

3.8 Derivatives of Inverse Functions and Inverse Trig Functions (Short Version)

Th: The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval (a,b). If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Or put in another way:

If $g(x) = f^{-1}(x)$, then $g'(x) = \dfrac{1}{f'(g(x))}$ for $f'(g(x)) \neq 0$ $\qquad f^{-1}(x)' = \dfrac{1}{f'(f^{-1}(x))}$

Ex. Find $(f^{-1})'(x)$ if

1.) $f(x) = x^3 - 2x - 1$ at $x = 3$

$f^{-1}(3) = ?$

where is x when $f(x) = 3$

$x^3 - 2x - 1 = 3$

$x^3 - 2x - 4 = 0$ (graph) $\to x = 2$

$f^{-1}(x)' = \dfrac{1}{3x^2 - 2}$

$= \dfrac{1}{3(2)^2 - 2}$

$= \boxed{\dfrac{1}{10}}$

2.) $f(x) = x^2$, for $x > 0$

$f'(x) = 2x$

$g(x) = f^{-1}(x) = \sqrt{x}$

$y = x^2$

$x = y^2$

$\pm\sqrt{x} = y$ $\quad x > 0$

$y = \sqrt{x}$

$\therefore$

$f^{-1}(x)' = \dfrac{1}{2x}$

$= \boxed{\dfrac{1}{2\sqrt{x}}}$

Th: The Derivative of the Inverse Trig Functions

$$\frac{d}{dx}\sin^{-1}u = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\csc^{-1}u = -\frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}\cos^{-1}u = -\frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}\sec^{-1}u = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}\tan^{-1}u = \frac{u'}{1+u^2} \qquad \frac{d}{dx}\cot^{-1}u = -\frac{u'}{1+u^2}$$

Ex. Find the derivative of:

1. $\sin^{-1} x^2$

$\dfrac{u'}{\sqrt{1-u^2}}$

$\dfrac{d}{dx}(\sin^{-1}x^2) = \dfrac{2x}{\sqrt{1-(x^2)^2}}$

$= \boxed{\dfrac{2x}{\sqrt{1-x^4}}}$

2. $\sin^{-1}(3x-1)$

$\dfrac{d}{dx}\sin^{-1}(3x-1) = \dfrac{3}{\sqrt{1-(3x-1)^2}}$

$= \dfrac{3}{\sqrt{1-(9x^2-6x+1)}}$

$= \dfrac{3}{\sqrt{6x-9x^2}}$

Ex. A particle is moving along the x-axis so that its position at any time $t \geq 0$ is $s(t) = \tan^{-1} 2t$. What is the velocity of the particle at $t = 4$?

$\frac{u'}{1+u^2}$

$$V(t) = s'(t) = \frac{2}{1+(2t)^2} = \frac{2}{1+4t^2}$$

$$@\underline{t=4}: \ v(4) = \frac{2}{1+4(4)^2} = \boxed{\frac{2}{65}}$$

ex. $\frac{d}{dx}\sec^{-1}(3x^2) = \frac{6x}{|3x^2|\sqrt{(3x^2)^2-1}}$

$\frac{u'}{|u|\sqrt{u^2-1}}$

$|3x^2|$ always +

$$= \frac{\cancel{6x}^{2}}{\cancel{3x^2}\sqrt{9x^4-1}}$$

$$= \boxed{\frac{2}{x\sqrt{9x^4-1}}}$$

ex. Find an equation of the line tangent to $y = \cot^{-1} x$ at $x = -1$

$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$

$$m = y' = -\frac{1}{1+x^2} \quad @\underline{x=-1}: \ y' = -\frac{1}{2} = m_T$$

$$y = \cot^{-1}(-1) = \frac{\pi}{2} - \tan^{-1}(-1) = \frac{\pi}{2} - -\frac{\pi}{4} = \frac{3\pi}{4}$$

$$\boxed{y - \frac{3\pi}{4} = -\frac{1}{2}(x+1)}$$

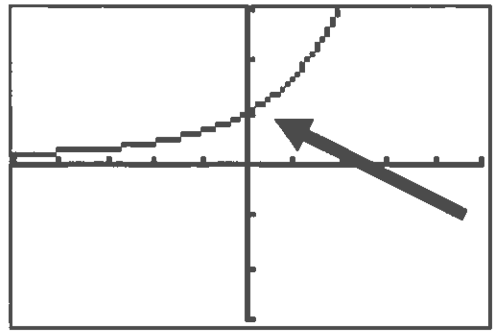
Homework: Pg. 170-171 #1 – 27o

Worksheet: 3.7-3.8 (Don't if you do Worksheet 3.7)

Quiz: Derivatives IV

Quiz: Impl Diff/Inv. Trig

3.9 Derivatives of Exponential and Logarithmic Functions



Property: $\lim_{h\to 0}\frac{e^h-1}{h}=1$

This can be proven using the graph of $y=\frac{e^x-1}{x}$

The Derivative of the Exponential Function

$\frac{d}{dx}(e^x)=e^x$ and $\frac{d}{dx}(e^u)=e^u\cdot u'$

Ex. Find the derivative of each of the following:

1. $y=5e^{3x}$

$y'=15e^{3x}$

2. $y=xe^x$ $\boxed{\text{Product Rule}}$

$y'=xe^x+e^x\cdot 1$

$y'=xe^x+e^x$

3. $y=e^{\cos x}$

$y'=e^{\cos x}\cdot(-\sin x)$

$y'=-e^{\cos x}\sin x$

4. $y=e^{x^2-2x}$

$y'=e^{x^2-2x}(2x-2)$

Ex. If $y=xe^{\sin x}$, then find y''

$y'=xe^{\sin x}\cos x+e^{\sin x}$

$y''=x\left[e^{\sin x}(-\sin x)+\cos x\cdot e^{\sin x}\cos x\right]+e^{\sin x}\cos x\cdot 1+e^{\sin x}\cos x$

$y''=-xe^{\sin x}\sin x+xe^{\sin x}\cos^2x+2e^{\sin x}\cos x$

Derivative of a^x:

For $a>0$ and $a\neq 1$,

$\frac{d}{dx}(a^x)=a^x\ln a$

Theorem: For $a>0$ and $a\neq 1$,

$\frac{d}{dx}(a^u)=a^u\ln a\cdot u'$

ex. At what point on the graph $y=2^x-3$ does the tangent line have a slope of 21?

$\ln\frac{a}{b}$
$\downarrow$
$\ln a-\ln b$

$y'=2^x\ln 2=21$

$2^x=\frac{21}{\ln 2}$

$x\ln 2=\ln 21-\ln(\ln 2)$

$x=\frac{\ln 21-\ln(\ln 2)}{\ln 2}$

$x=4.4172796$

$y=18.3665$

$(4.417, 18.3665)$

Derivative of ln x:

Theorem: $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$ or $\frac{u'}{u}$

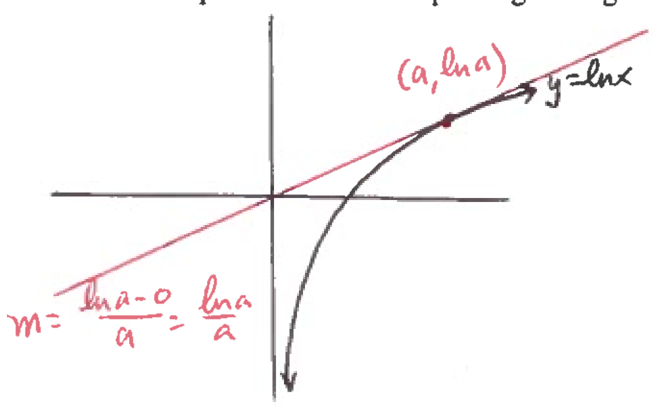
Ex. Find the equation of the line tangent to $y = \ln x$ at $x = 4$

$y' = \frac{1}{x}$ $\quad$ @$x=4$: $y' = m_T = \frac{1}{4}$

$y = \ln 4$

$\therefore \boxed{y - \ln 4 = \frac{1}{4}(x-4)}$

Ex Find the equation of the line passing through the origin and tangent to $y = \ln x$.



slope from line

$m_T = y' = \frac{1}{x}$

@$x=a$: $y' = \frac{1}{a} = \frac{\ln a}{a}$

$\therefore \ln a = 1$

$a = e$

$(e, \ln e)$

$(e, 1)$

$\boxed{y - 1 = \frac{1}{e}(x-e)}$

Derivative of $\log_a x$:

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\cdot\frac{du}{dx}$$

$\frac{u'}{u\ln a}$

Ex. Find y' if $y = \log_a a^{\sin x}$

$y' = \frac{\cancel{a^{\sin x}}\cdot\cancel{\ln a}\cos x}{\cancel{a^{\sin x}}\cancel{\ln a}} = \boxed{\cos x}$??

$\log_a a^x = x$

$\therefore \log_a a^{\sin x} = \sin x$

$\frac{d}{dx}\sin x = \underline{\cos x}$

Logarithmic Differentiation: Sometimes you can use the properties of logs to help simplify before find the derivative.

Ex. Find y' if $y = x^x, x > 0$

You can not use any of the rules/properties given because of the x's in both the base and the exponent. In this case, we will use logs (natural logs):

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{d}{dx} \ln y = \frac{d}{dx}(x \ln x)$

$\frac{1}{y} \cdot \frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = y(1+\ln x)$

$\boxed{\frac{dy}{dx} = x^x(1+\ln x)}$

ex $y = x^{\cos x}$

$\ln y = (\cos x)\ln x$

$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \frac{1}{x} + \ln x(-\sin x)$

$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{x} - (\sin x)\ln x$

$\frac{dy}{dx} = y\left[\frac{\cos x}{x} - (\sin x)\ln x\right]$

$\boxed{\frac{dy}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - (\sin x)\ln x\right]}$

ex $y = x^{\ln x}$ ← on HW

$\ln y = \ln x \cdot \ln x$

$\frac{1}{y} \cdot y' = \ln x \cdot \frac{1}{x} + \ln x \cdot \frac{1}{x}$

$y' = y\left[\frac{2\ln x}{x}\right]$

$\boxed{y' = x^{\ln x}\left(\frac{2\ln x}{x}\right)}$

ex $y = (\sin x)^{\cos x}$

$\ln y = (\cos x)\ln(\sin x)$

$\frac{1}{y} y' = \cos x \cdot \frac{\cos x}{\sin x} + \ln(\sin x) \cdot (-\sin x)$

$y' = y\left[\frac{\cos^2 x}{\sin x} - (\sin x)\ln(\sin x)\right]$

$\boxed{y' = (\sin x)^{\cos x}\left[\frac{\cos^2 x}{\sin x} - (\sin x)\ln(\sin x)\right]}$

Homework: Pg. 178-179 #3-39 (3's), 41, 43, 44, 47, 51, 52

Worksheet: 3.7-3.9

Day 2: p.178-179

Quiz: Derivative of e^x, $\ln x$, ..

Quiz: 3.7-3.9

Formula Quiz!