Chapter 3 – Derivatives

3.1 Derivative of a Function

Def: The Formal Definition of the Derivative

The ***derivative*** of the function $f(x)$ with respect to the variable x is the function $f'(x)$, pronounced "*f prime of x*", whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Note: From last chapter, this is the slope of the curve or the slope of the tangent to the curve.

- The domain of f', the set of points in the domain of f for which the limit exists, may be smaller than the domain of f.
- If $f'(x)$ exists, we say that f has a derivative (is differentiable) at x.
  - If a function is differentiable at every point of its domain, then it is called a differentiable function.

Ex. Differentiate (find the derivative of) $f(x) = x^3$

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= \lim_{h \to 0} 3x^2 + 3xh + h^2 \\
&= 3x^2
\end{aligned}$$

Def: Derivative at a Point (Alternate Definition)

The ***derivative*** of the function $f(x)$ at the point $x = a$ is the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided the limit exists.

Ex. Differentiate $f(x) = \sqrt{x}$ using the alternate definition

$$
\begin{aligned}
f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \\
&= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \quad \leftarrow \text{Multiply by conjugate – rationalize} \\
&= \lim_{x \to a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} \\
&= \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} \\
&= \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}
\end{aligned}
$$

Notation:

There are many ways to denote the derivative of a function $y = f(x)$. Other common notations are

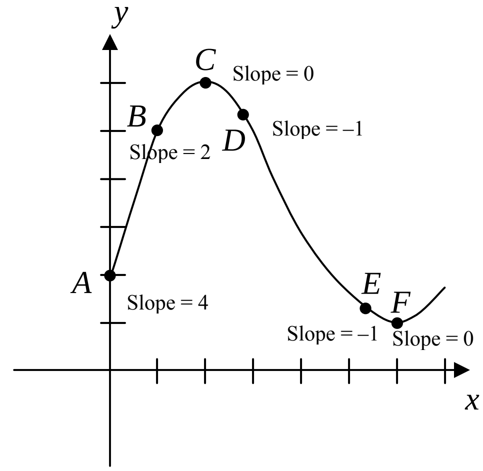
y'	"*y* prime"	Handy – but does not name the independent variable (x)
$\frac{dy}{dx}$	"*dy dx*" "the derivative of *y* with respect to *x*"	Names both variables and uses *d* for derivative
$\frac{df}{dx}$	"*df dx*" "the derivative of *f* with respect to *x*"	Emphasizes the function's name
$\frac{d}{dx} f(x)$	"*d dx* of *f* at *x*" "the derivative of *f* at *x*"	Emphasizes the idea that differentiation is an operation performed on *f*

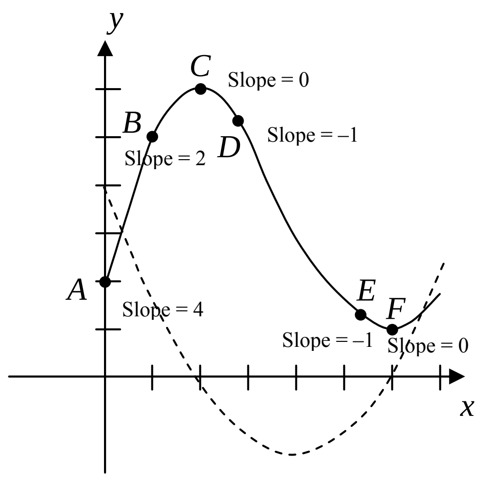
Homework: Pg 101 – 104 #1–6, 11, 12, 15

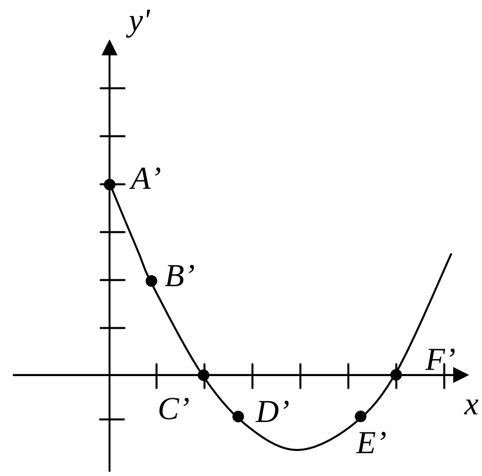
3.1 The Derivative (Day 2)

<u>Graphing *f'* from *f*</u>:

- Suppose we wanted to graph the function $y = f'(x)$ based on the graph of $y = f(x)$ to the right.
- Basically, plot a new point $(a, f'(a))$ on another set of axes. This will be the graph of $y = f'(x)$.
    - o After plotting each point (the *x* coordinate with its slope of tangent line (y'), we draw a smooth curve through the points.
- If we superimpose the derivative function onto the original graph, we get the graph below. (*f'* is the dashed line).



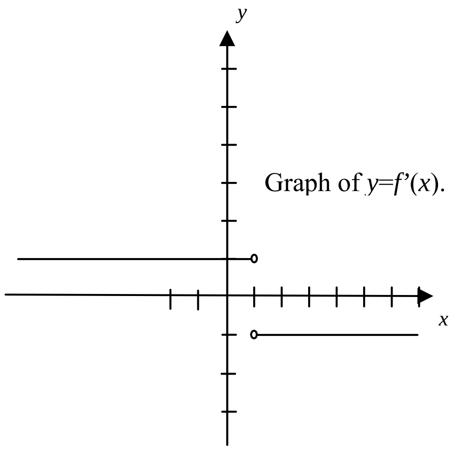


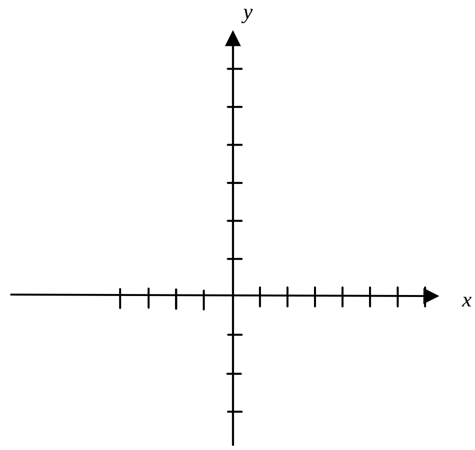


- Notice the different "sections" of the two curves:
    - o From A to C, $f(x)$ is increasing. What do you notice about $f'(x)$ on the same domain values of *x?*
    - o From C to F, $f(x)$ is decreasing. What do you notice about $f'(x)$ on the same domain values of *x*?
    - o What can you say about the slope of an increasing function? A decreasing function?
    - o What is the slope of the tangent at C and F? What do you notice happens with $y = f(x)$ around these two points?

On the set of axes below, sketch the graph of a function f that has the following properties:

(i) $f(1) = 4$
(ii) the graph of f', the derivative of f, is shown below to the left
(iii) f is continuous for all x.





- What do you notice about the $y = f(x)$ at $x = 1$? What do you notice about $y = f'(x)$ at $x = 1$?

<u>One-Sided Derivatives</u>:

- A function $y = f(x)$ is <u>differentiable on a closed interval [a,b]</u> if it has a derivative at every interior point on the interval and if the limits:

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} \text{ [the right-hand derivative at } a\text{] or } \lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$$

$$\lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h} \text{ [the left-hand derivative at } b\text{] or } \lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$$

exist at the endpoints

ex. Show the following function has left-hand and right-hand derivatives at $x = 0$, but no derivative there.

$$f(x)=\begin{cases} x^2, x\le 0 \\ 2x, x>0 \end{cases}$$

Homework: pg. 101-104 #7-10, 14, 16, 18

3.2 Differentiability

Ex. Find the derivative for each of the following:

1. $f(x) = \sqrt{x}$

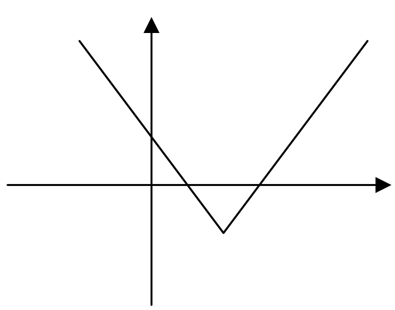
2. $f(t) = \frac{2}{t}$

3. $f(x) = 2x^2 + x - 1$

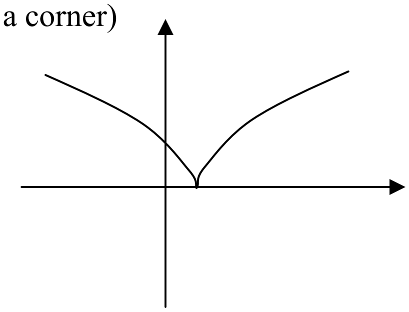
4. $f(x) = x + \frac{1}{x}$

Property: $y = f(x)$ will have no derivative at $x = a$ where the graph has:

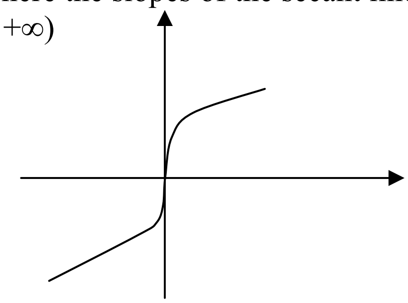
1. A corner: One-sided derivatives differ



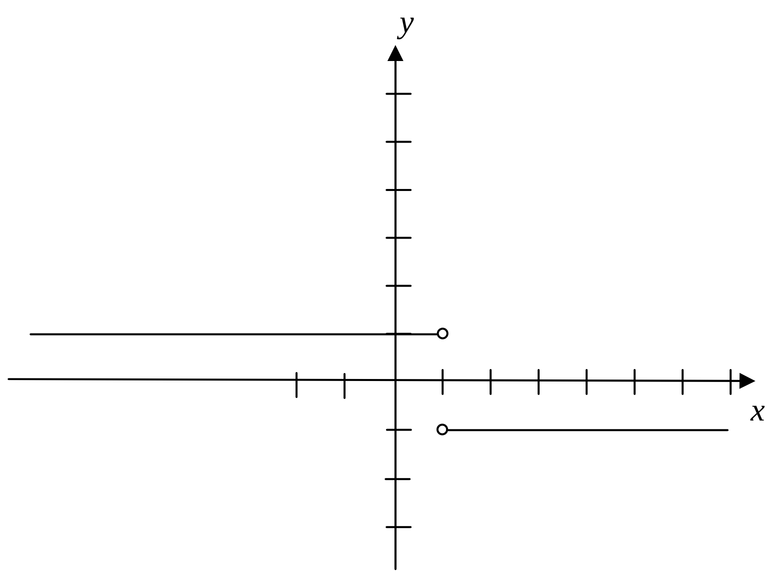
2. <u>a cusp</u>: where the slopes of the secant lines approach ∞ from one side and –∞ from the other side (an extreme case of a corner)



3. <u>a vertical tangent</u>: where the slopes of the secant lines approach either +∞ or –∞ from both sides (this example: +∞)



4. <u>A point of discontinuity</u>:



<u>To find where a function is not differentiable</u>:

- If it has an absolute value, then set the inside of the absolute value equal to 0 and solve for *x*

- Most functions in calculus will be differentiable, which means *no corners, no cusps,no points of discontinuity,* or *vertical tangent lines* within their domains. Their curves will be smooth with a well-defined slope at each point.

Differentiability Implies Continuity

Theorem: If f has a derivative at $x = a$, then f is continuous at $x = a$.

Proof: We will show that $\lim_{x \to a} f(x) = f(a)$ or better yet: $\lim_{x \to a}[f(x) - f(a)] = 0$

$$
\begin{aligned}
\lim_{x \to a}[f(x) - f(a)] &= \lim_{x \to a}\left[(f(x) - f(a)) \cdot \frac{x-a}{x-a}\right] \\
&= \lim_{x \to a}\left[(x-a) \cdot \frac{f(x) - f(a)}{x-a}\right] \\
&= \lim_{x \to a}(x-a) \cdot \lim_{x \to a}\frac{f(x) - f(a)}{x-a} \\
&= 0 \cdot f'(a) \\
&= 0
\end{aligned}
$$

This is by definition, $f'(a)$

Although Differentiability implies Continuity, the converse (Continuity implies Differentiability) is not true.

Ex. $f(x) = |x - 2|$

To summarize the relationship between Differentiability and Continuity:

1. If a function is differentiable at $x = a$, then it is continuous at $x = a$
2. If a function is continuous at $x = a$, then it does not have to be differentiable at $x = a$
3. If a function is NOT continuous at $x = a$, then it does NOT have a derivative at $x = a$.

Derivatives with Calculators

You can find the derivative of a function at a point using your TI-83+ using the "nderiv" function. The command is:

nderiv($f(x)$, x, a)

ex. nderiv(x^2,x, 4) → 8
nderiv(sin x, x, $\pi/4$) → 0.7071

You are only to use this to check your answers or if the questions instructs you to use it (NDER in book)

Homework: Pg. 111 #1–10, 17

3.3 Rules for Differentiation

Rule #1: Derivative of a Constant Function

If f is the function with the constant value c, $f(x) = c$, then

$$\frac{df}{dx} = f'(x) = 0$$

Proof:

Property: $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + ... + ab^{n-2} + b^{n-1})$

Rule #2: Power Rule for Positive Integer Power of x

If $f(x) = x^n$ and n is a positive integer, then

$$\frac{df}{dx} = f'(x) = nx^{n-1}$$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{(x+h-x)[(x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + ... + (x+h)x^{n-2} + x^{n-1}]}{h} \\ &= \lim_{h\to 0} \frac{h[(x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + ... + (x+h)x^{n-2} + x^{n-1}]}{h} \\ &= \lim_{h\to 0} (x+h)^{n-1} + (x+h)^{n-2}x + (x+h)^{n-3}x^2 + ... + (x+h)x^{n-2} + x^{n-1} \\ &= x^{n-1} + x^{n-1} + x^{n-1} + ... + x^{n-1} + x^{n-1} \quad \longleftarrow \text{There are } n \text{ identical terms of } x^{n-1} \\ &= nx^{n-1} \end{aligned}$$

To differentiate x^n, multiply the exponent n by x to 1 less than the n.

Ex. Find the derivative of

1. $f(x) = x^5$
2. $f(x) = x^2$
3. $f(x) = x^{12}$

Rule #3: The Constant Multiple Rule

If f is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cf) = c\frac{df}{dx}$$

Proof:

$$\frac{d}{dx}(cf) = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$
$$= \lim_{h\to o}\frac{c[f(x+h)-f(x)]}{h}$$
$$= c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= c\cdot\frac{df}{dx}$$

Rule #4: The Sum and Difference Rule

If u and v are differentiable functions of x, then their sum and difference are differentiable at every point where u and v are differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

Proof: Let $f(x) = u(x) + v(x)$

Homework: Pg. 120 #1–9, 20, 25, 27

3.3 Rules for Differentiation (Day 2)

Rule #5: The Product Rule

The product of two differentiable functions *u* and *v* is differentiable, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Alternate Writing: If $f(x) = u(x) \cdot v(x)$, then $f'(x) = u(x)v'(x) + v(x)u'(x)$

Proof:

Ex. Find $f'(x)$ if $f(x) = (x^2 + 1)(x^3 + 3)$

Ex. Find $f'(x)$ if $f(x) = (x^3 + 3x)(2x^2 + 3x + 3)$

Rule #6: The Quotient Rule

At a point where $v \neq 0$, the quotient $y = \frac{u}{v}$ of two differentiable function is differentiable, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Alternate Writing: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$

Proof:

Ex. Differentiate $f(x) = \frac{x^2 - 1}{x^2 + 1}$.

Ex. Write the equation of the line tangent to $f(x) = \frac{x}{x-1}$ at $x = 2$

Rule #7: Power Rule for Negative Integer Power of x

If $f(x) = x^n$ and n is a negative integer and $x \neq 0$, then

$$\frac{df}{dx} = f'(x) = nx^{n-1}$$

Proof:

Ex. Find the equation for the line tangent to the curve $y = \dfrac{x^2+3}{2x}$

Using the Quotient Rule

Using the Power Rule

Higher Order Derivatives

- The derivative $y' = \dfrac{dy}{dx}$ is called the *first derivative* of y with respect to x. If the first derivative is also a differentiable function, then

  $$y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$ is called the *second derivative of y with respect to x*.

  y'' is pronounced "y double–prime"
- y''' is the third derivative of y with respect to x

  $$y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

- Continuing, we end up with

$$y^{(n)} = \frac{d}{dx} y^{(n-1)} \quad \text{The } n^{th} \textit{ derivative of y with respect to x } \left(\frac{d^n y}{dx^n}\right)$$

Ex. Find the first 4 derivatives for $f(x) = x^4 + 2x^3 - 5x^2 + 7x - 10$

Examples:

Homework: Pg 120 – 121 #10–19, 22, 23, 24, 31, 33, 34

3.4 Velocity and Other Rates of Change

Def: *Instantaneous Rates of Change*

The *Instantaneous Rate of Change* of f with respect to x at a is the derivative.

$$f'(a) = \lim_{x \to h} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Ex. Find the rate of change of the area A of a circle with respect to its radius r. What is the rate of change of A when $r = 3$, $r = 6$, and when $r = 8$?

Motion along a Line

Def: Suppose that an object is moving along a coordinate line (the x-axis) so that we know its position, called s, on that line as a function of time t.

$$s = f(t)$$

The *displacement* of the object over the time interval t to $t + \Delta t$ is

$$\Delta s = f(t+\Delta t) - f(t)$$

Def: The *average velocity* of an object over that time interval is

$$v_{avg} = \frac{displacement}{time} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Def: The *instantaneous velocity* is the velocity of the object at the exact instant t.

This can be found by taking the limit of the average velocity as $\Delta t \to 0$.

Therefore, the velocity of an object at time t is the derivative of the position function $s = f(t)$ with respect to time. At time t, the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

ex. A particle is moving along a line such that its position is given by the function $s(t) = 4t^3 + 3t^2 - 12t$. What is the velocity of the particle at t =1, 2, 5, and 6 seconds.

Def: Speed is the absolute value of velocity.

$$Speed = | v(t) |$$

The difference between speed and velocity is that velocity gives you two values: how fast the object is moving and in which direction. Speed does not.

Def: *Acceleration* measures how quickly an object picks up or loses speed. Acceleration is the derivative of velocity with respect to time. If a body's velocity at time t is $v(t) = \frac{ds}{dt}$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

Note: Acceleration is the position function's 2nd Derivative.

Ex. Using the previous example, what is the acceleration of the particle at t = 1, 2, 5, and 6?

Some Constants to Know About the Earth:

Experimental and theoretical investigations revealed that the distance a body released from rest (v = 0) falls freely is proportional to the square of the amount of time it has fallen.

$s = \frac{1}{2}gt^2$, where s is distance, g is acceleration due to gravity, and t is time. Units are determined but units used to measure s and t

Free–Fall Constants:

English Units: $g = 32\,\frac{ft}{\sec^2}$, $s = \frac{1}{2}(32)t^2 = 16t^2$ (feet)

Metric Units: $g = 9.8\,\frac{m}{\sec^2}$, $s = \frac{1}{2}(9.8)t^2 = 4.9t^2$ (meters)

Units for acceleration would be $ft/\sec^2$ is read "feet per second squared"

Ex. A dynamite blast propels a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph). It reaches a height of $s = 160t - 16t^2$ feet after t seconds.

(a) How high does the rock go?
(b) What is the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?
(c) What is acceleration of the rock at any time t during its flight (after the blast)?
(d) What does the rock hit the ground?

Homework: Pg. 129-130 #1-6, 9, 12, 13

3.4b Marginal Cost and Revenue (Economics)

- Engineers use velocity and acceleration to refer to the derivatives of functions describing motion.
- Economists use derivatives as well for rates of change. This is referred to as *marginals*.
- In manufacturing:
  - The cost of production $c(x)$ is a function of x, the number of units produced.
  - The *marginal cost of production* is the rate of change of the cost with respect to the level of production.
    - In other words: $\frac{dc}{dx}$ or $c'(x)$
  - Revenue is the amount of money made from the selling of a product. It is a function of the number of units sold, x. $r(x)$
  - *Marginal Revenue* is the rate of change of the revenue with respect to the level of production, or $r'(x)$

Ex. Suppose the cost to produce x radiators when 8 to 10 radiators are produced is given by $c(x) = x^3 - 6x^2 + 15x$ and the dollars of revenue from selling these x radiators is $r(x) = x^3 - 3x^2 + 12x$. If the shop produces 10 radiators per day, find the marginal cost and marginal revenue.

Homework: Pg. 130-131 #8, 10, 11, 14, 20

3.5 Derivatives of Trigonometric Functions

Using the Formal Definition of the Derivative, find $f'(x)$ if $f(x) = \sin x$

Therefore: The derivative of $f(x) = \sin x$ is ________________

Using the Formal Definition of the Derivative, find $f'(x)$ if $f(x) = \cos x$

Therefore: The derivative of $f(x) = \cos x$ is ________________

Ex. Find the derivative of each of the following.

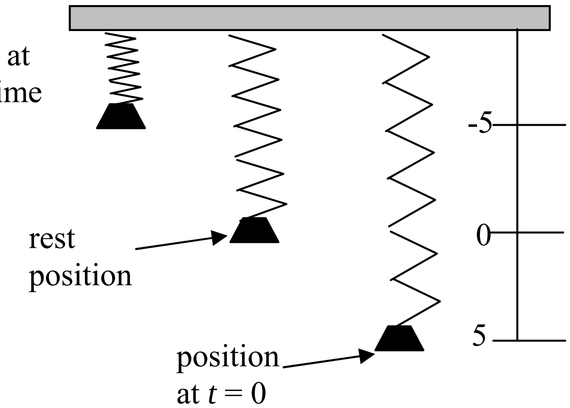
1. $y = x^2 \sin x$
2. $f(x) = \dfrac{\cos x}{1 - \sin x}$

Def: The motion of a weight bobbing up and down at the end of a spring is an example of *simple harmonic motion*.

ex. A weight is hanging from a spring (see diagram right) is stretched 5 units beyond its rest point (s=0) and released at time t = 0 to bob up and down. Its position at any later time t is

$$s = 5\cos t$$

What is the velocity and acceleration at time t?



Here is what we get out of the equations:

1. As time passes, the weight moves down and up between –5 and 5 on the axis. The amplitude of the motion is 5. The frequency is 1, therefore the period is 2π.
2. The velocity is $v = -5\sin t$. Its greatest magnitude is 5, therefore the maximum velocity the weight get to is 5 (when $\cos t = 0$). It is not moving when $\sin t = 0$.
3. The acceleration is always the direct opposite of the position ($-5\cos t$). When the weight is above the rest position, it is slowing down (due to gravity), when it is below rest position, the spring is pulling up (due to tension).
4. The acceleration, $a = -5\cos t$, is 0 only at the rest position ($\cos t = 0$) and the force of gravity and the force from the spring offset each other.
   - When the weight is anywhere else, the two forces (gravity and the spring) are unequal and acceleration is not 0.
   - Acceleration is greatest when the weight is furthest from rest position ($\cos t = \pm 1$)

Def: Jerk is the derivative of acceleration. If a body's position at time t is $s(t)$, the body's jerk at time t is:

$$j(t) = \frac{da}{dt} = \frac{d^3 s}{dt^3} = s'''(t)$$

ex. What is the jerk force on the weight in the previous example at time t? When is the jerk force at its highest?

Derivatives of Other Trig Functions:

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

ex. Find $f'(x)$ for each of the following:

1. $f(x) = \dfrac{\tan x}{x}$
2. $f(x) = \sin x \tan x$

Ex. Find y'' if $y = \sec x$

Ex. Find y' if $y = \sin 2x$ (hint use formula for $y = \sin 2x$)

Homework: Day 1: Pg 140 #1–23o, 27, 29
Day 2: Pg 140 #2-26e, 30

3.6 Chain Rule

recall: A composite function is defined as a function containing another function.

$$f(g(x)) = f[g(x)] = (f \circ g)(x)$$

Suppose you have the function $y = 2(3x - 5)$ and you were asked to find the derivative.

The function $y = 2(3x - 5)$ is a composite of the functions $y=2u$ and $u = 3x - 5$

Distributing the 2 gets $y = 6x - 10$

Suppose we take the derivative of all three equations:

$y = 6x - 10$	$y=2u$	$u = 3x - 5$
$\frac{dy}{dx} = 6$	$\frac{dy}{du} = 2$	$\frac{du}{dx} = 3$

See a relationship between the numbers? $6 = 2 \times 3$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Does this always work??

Ex. Find the derivative of $y = (3x^2 + 1)^2$

$y = u^2$ $\qquad u = 3x^2 + 1$

$\frac{dy}{du} = 2u$ $\qquad \frac{du}{dx} = 6x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= 2u \cdot 6x$$
$$= 2(3x^2 + 1)(6x)$$
$$= (6x^2 + 2)(6x)$$
$$= 36x^3 + 12x$$

$$y = (3x^2 + 1)^2$$
$$y = (3x^2 + 1)(3x^2 + 1)$$
$$y = 9x^4 + 6x^2 + 1$$
$$\frac{dy}{dx} = 36x^3 + 12x$$

So it looks like it works………

Rule #8: The Chain Rule

If f is differentiable at the point $u = g(x)$ and g is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then : $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Ex. An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$.
. Find the velocity of the object as a function of t.

Ex. Find the derivative of $f(t) = \tan(5 - \sin 2t)$

Power Chain Rule: If $y = [f(x)]^n$, then $y' = n[f(x)]^{n-1} f'(x)$

Power Chain Rule (alternate form): $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

<u>One more look at it</u>

If $f(x) = u^n$, where u is a differentiable function of x, then :

$$f'(x) = nu^{n-1} \cdot u'$$

Ex. Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \frac{\pi}{3}$.

Ex. Show the slope of every line tangent to the curve $y = \frac{1}{(1-2x)^3}$ is positive.

Homework: Pg. 146 – 147 #1-19o, 26, 31

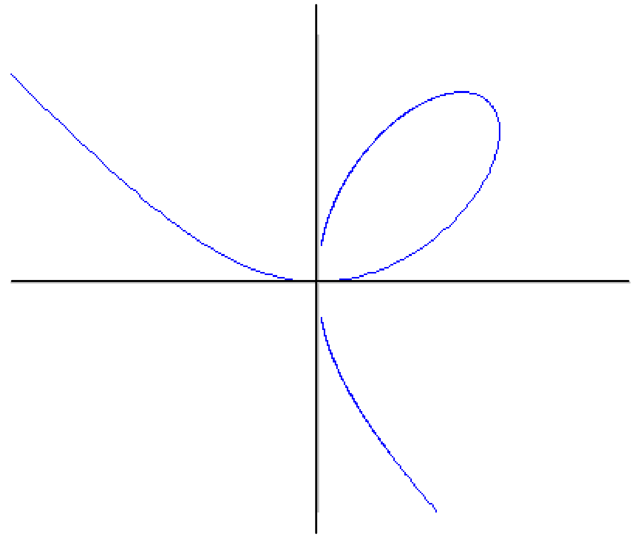
3.7 Implicit Differentiation

To the right is a curve called a *folium*

- First discussed by Descartes in 1638
- The equation of the curve is:

$$x^3 + y^3 - 9xy = 0$$

- But this equation can not be solved for *y* so finding the slope of the tangent at any point can not be done the way we have before.
- In previous sections, we always had an equation of the form $y = f(x)$
    - This kind of differentiation is called *explicit differentiation*.



Consider the equation $xy = 1$

To find $\frac{dy}{dx}$, we would rewrite the above equation as

$$y = \frac{1}{x} \text{ or}$$

$$y = x^{-1}$$

Finding the derivative: $\frac{dy}{dx} = -1x^{-2} = \frac{-1}{x^2}$

There is another way to do this. This is called *implicit differentiation*.

We can differentiate (take the derivative) of both sides of $xy = 1$ before solving for *y*.

This is the product rule:
(*uv*)' = *uv*'+ *v u*'

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$$

$$x\frac{dy}{dx} + y(1) = 0$$

$$x\frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

Substituting $y = \frac{1}{x}$ into *y*:

$$\frac{dy}{dx} = -\frac{\frac{1}{x}}{x} = -\frac{\frac{1}{x}}{x} \cdot \frac{x}{x} = -\frac{1}{x^2}$$

This agrees with the previous result using explicit differentiation.

- The advantage of implicit differentiation is that you don't have to solve the equation for *y*.
- So in general, you do not have to substitute back in for *y*. Leave the *y* in the equation unless otherwise stated.
- Use implicit differentiation if you can't solve for *y*.

Ex. Find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$

Differentiating both sides, we get:

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$

Chain Rule

$$5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$

$$5(2y)\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

$$10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

Factor out $\frac{dy}{dx}$

$$\frac{dy}{dx}(10y + \cos y) = 2x$$

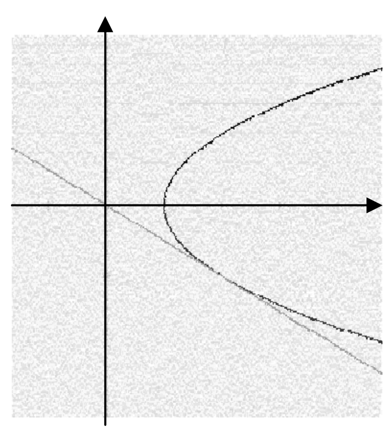
$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

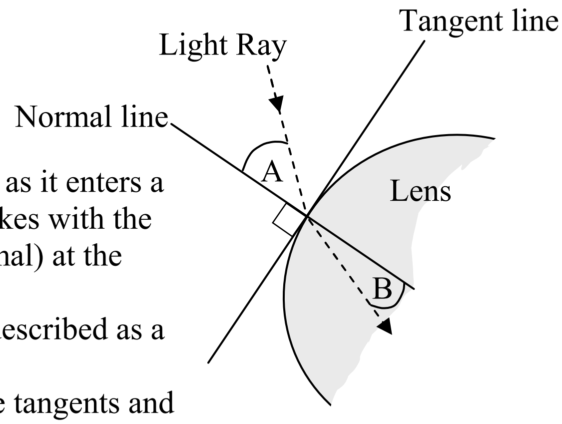
Implicit Differentiation Process:

1. Differentiate both sides of the equation with respect to *x*
2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation
3. Factor out $\frac{dy}{dx}$
4. Solve for $\frac{dy}{dx}$

ex. Find the slope of the tangent line at (4,0) to the graph of $7y^4 + x^3y + x = 4$.

Ex. Find the slope of the tangent line at (2,–1) to $y^2 - x + 1 = 0$





Lenses, Tangents, and Normal Lines

- In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens (the normal) at the point of entry.
- How a lens is curved (the profile of lenses) is often described as a quadratic curve.
  - We can use implicit differentiation to find the tangents and normals.

Ex. Find the normal and tangent to the ellipse $x^2 - xy + y^2 = 7$ at the point (–1,2).

Derivatives of Higher Order

- Implicit differentiation can be used to find derivatives of higher order (2nd, 3rd, etc…)

Ex. Use implicit differentiation to find $\frac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$

Rational Powers of Differentiable Functions (Fractional Exponents)

- We know the Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$ is true for any integer *n* (Rules 2 and 7).
- We can now prove it true for any *n*:

> **<u>Rule #9</u>: Power Rule for Rational Powers of *x***
> If *n* is any rational number, then:
> $$\frac{d}{dx}x^n = nx^{n-1}$$
> If $n < 1$, then the derivative does not exist at $x = 0$.

Proof:

Let *p* and *q* be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{\frac{p}{q}}$, then

$$y^q = x^p$$

Since *p* and *q* are integers, we can use the Power Rule to differentiate both side with respect to *x*

$$y^q = x^p$$

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

$$\frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}}$$

$$= \frac{p}{q}\cdot\frac{x^{p-1}}{y^{q-1}}$$

$$= \frac{p}{q}\cdot\frac{x^{p-1}}{(x^{p/q})^{q-1}}$$

Substituting what *y* equals

"<u>Aside</u>:"
$$(x^{\frac{p}{q}})^{q-1} = x^{\frac{p}{q}\cdot(q-1)}$$
$$= x^{p-\frac{p}{q}}$$

$$= \frac{p}{q}\cdot\frac{x^{p-1}}{x^{p-p/q}}$$

$$= \frac{p}{q}\cdot x^{(p-1)-(p-p/q)}$$

Property of exponents

$$= \frac{p}{q}x^{p-1-p+p/q}$$

$$= \frac{p}{q}x^{\frac{p}{q}-1}$$

So this Rule works for ANY kind of exponent!

Ex. Find the derivative of each of the following:

1. $y = \sqrt{x}$

2. $y = \sqrt[3]{x^2}$

3. $y = (\cos x)^{-\frac{1}{5}}$

Homework: **Day 1: Pg. 155 #1-19odd, 23-35 odd, 37(a)**
Day 2: Pg. 155 – 156 #2, 6, 10, 14, 16, 20, 24, 26, 28, 32, 36, 38(a), 41, 42, 43

3.8 Derivatives of Inverse Functions and Inverse Trig Functions

__Th__: If f is differentiable at every point of an interval I and ${}^{df}\!/_{dx}$ is never zero on I, then f has an inverse and f^{-1} is differentiable at every point of the interval $f(I)$.

Recall: By definition: $f(f^{-1}(x)) = x$

Suppose we take the derivative of both sides of this equation:

$$\frac{d}{dx} f(f^{-1}(x)) = 1$$ *using the chain rule on the left side, we get:*

$$f'(f^{-1}(x)) \cdot f^{-1}(x)' = 1$$

$$f^{-1}(x)' = \frac{1}{f'(f^{-1}(x))}$$

This leads to the following theorem:

__Th__: **The Derivative of an Inverse Function**

Let f be a function that is differentiable on an interval (a,b). If f has an inverse function g. then g is differentiable at any x for which $f'(g(x)) \neq 0$. Or put in another way:

If $g(x) = f^{-1}(x)$, then $g'(x) = \dfrac{1}{f'(g(x))}$ for $f'(g(x)) \neq 0$

Ex. Find $(f^{-1})'(x)$ if

1.) $f(x) = x^3 - 2x - 1$ at $x = 3$

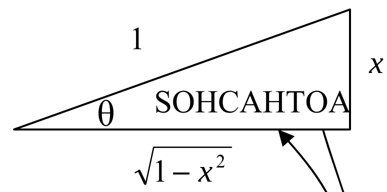
2.) $f(x) = x^2$, for $x > 0$

Derivative of $y = \sin^{-1} u$ (Arcsin):

Recall: A function can have an inverse if it is one-to-one.

- We know that $y = \sin x$ is differentiable for all values of x.
  - We need to use a part of the curve so that it is one-to-one. This part is on the open $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
    - By not including $\pm\frac{\pi}{2}$, we can avoid part of the theorem above about the derivative being equal to 0.

$$y = \sin x$$
$$y' = \cos x$$
$$0 = \cos x$$
$$\pm\frac{\pi}{2} = x$$



  - So let's work on the derivative of $y = \sin^{-1} x$.

$$y = \sin^{-1} x$$
$$x = \sin y$$
$$\frac{d}{dx}x = \frac{d}{dx}(\sin y)$$
$$1 = \cos y \frac{dy}{dx}$$
$$\frac{1}{\cos y} = \frac{dy}{dx}$$
$$\frac{1}{\sqrt{1-x^2}} = \frac{dy}{dx}$$

"<u>Aside</u>"
We can use the identity

$$\sin^2 y + \cos^2 y = 1$$
$$\cos^2 y = 1 - \sin^2 y$$
$$\cos y = \sqrt{1 - \sin^2 y}$$
$$\cos y = \sqrt{1 - x^2}$$

<u>Using the Inverse Th</u>:

$$f(x) = \sin x$$
$$g(x) = \sin^{-1} x$$
$$g'(x) = \frac{1}{f'(g(x))}$$
$$g'(x) = \frac{1}{\cos(\sin^{-1} x)}$$
$$g'(x) = \frac{1}{\sqrt{1-x^2}}$$

  - We can now rewrite this where u is a function of x:

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad \text{for } |u| < 1$$

Ex. Find the derivative of:

1. $\sin^{-1} x^2$
2. $\sin^{-1}(3x-1)$

Derivative of Tan^{-1} *u* (Arctan):

Using a similar approach so before, find the derivative:

$$y = \tan^{-1} x$$

$$\tan y = x$$

$$\frac{d}{dx}(\tan y) = \frac{d}{dx} x$$

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

Therefore, if *u* is a function of *x*, then: $\dfrac{d}{dx}\tan^{-1} u =$

Ex. A particle is moving along the *x*-axis so that its position at any time $t \geq 0$ is $s(t) = \tan^{-1} 2t$. What is the velocity of the particle at $t = 4$?

Derivative of the Arcsecant ($\sec^{-1}$ *u*):

$$y = \sec^{-1} x$$

$$x = \sec y$$

$$\frac{d}{dx} x = \frac{d}{dx}(\sec y)$$

$$1 = \sec y \tan y \frac{dy}{dx}$$

$$\frac{1}{\sec y \tan y} = \frac{dy}{dx}$$

$$\sec y = x$$

$$1 + \tan^2 y = \sec^2 y$$

$$\tan^2 y = \sec^2 y - 1$$

$$\tan y = \pm\sqrt{\sec^2 y - 1}$$

$$\tan y = \pm\sqrt{x^2 - 1}$$

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$

ex. $\frac{d}{dx}\sec^{-1}(3x^2) =$

Derivatives of the Other Trig Functions:

<u>Property</u>: Inverse Co-Function Identities

	Taking Derivative	We get:
$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$	$\frac{d}{dx}\cos^{-1} x = \frac{d}{dx}(\frac{\pi}{2} - \sin^{-1} x)$	$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$
$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$ →	$\frac{d}{dx}\cot^{-1} x = \frac{d}{dx}(\frac{\pi}{2} - \tan^{-1} x)$ →	$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$
$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$	$\frac{d}{dx}\csc^{-1} x = \frac{d}{dx}(\frac{\pi}{2} - \sec^{-1} x)$	$\frac{d}{dx}\csc^{-1} x = -\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$

ex. Find an equation of the line tangent to $y = \cot^{-1} x$ at $x = -1$

Summary: Here are the Formulas for the Inverse Trig Function Derivatives:

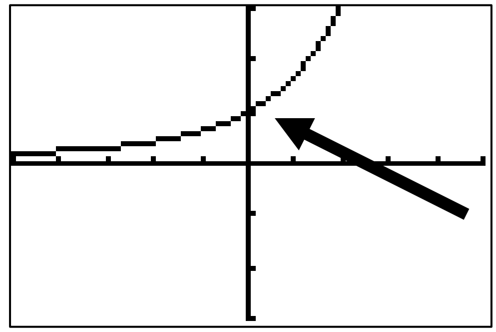
$$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\csc^{-1} u = -\frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \qquad \frac{d}{dx}\sec^{-1} u = \frac{1}{\lvert u\rvert\sqrt{u^2-1}}\frac{du}{dx}$$

$$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx} \qquad \frac{d}{dx}\cot^{-1} u = -\frac{1}{1+u^2}\frac{du}{dx}$$

Homework: Pg 162–163 #1–10, 11–23o

3.9 Derivatives of Exponential and Logarithmic Functions



Property: $\lim_{h\to 0}\frac{e^h-1}{h}=1$

This can be proven using the graph of $y=\frac{e^x-1}{x}$

This property will help in find the derivative of $y = e^x$:

So $\frac{d}{dx}(e^x)=$________ and $\frac{d}{dx}(e^u)=$______________

Ex. Find the derivative of each of the following:

1. $y=5e^{3x}$
2. $y=xe^x$
3. $y=e^{\cos x}$
4. $y=e^{x^2-2x}$

Ex. If $y=xe^{\sin x}$, then find y"

<u>Derivative of a^x</u>:

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}(a^x) = a^x \ln a$$

<u>Proof</u>:

$$a^x = e^{\ln a^x}$$
$$= e^{x \ln a}$$

Therefore:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x\ln a})$$
$$= e^{x\ln a} \cdot \frac{d}{dx}(x \ln a)$$
$$= e^{x \ln a}(\ln a)$$
$$= a^x \ln a$$

So…..

> **<u>Theorem</u>:** For $a > 0$ and $a \neq 1$,
> $$\frac{d}{dx}(a^u) = a^u \ln a \frac{du}{dx}$$

ex. At what point on the graph $y = 2^x - 3$ does the tangent line have a slope of 21?

$$\frac{d}{dx}(2^x - 3) = 2^x \ln 2 - 0$$
$$\frac{d}{dx}(2^x - 3) = 2^x \ln 2$$
$$21 = 2^x \ln 2$$
$$\frac{21}{\ln 2} = 2^x$$
$$\ln\left(\frac{21}{\ln 2}\right) = \ln 2^x$$
$$\ln 21 - \ln(\ln 2) = x \ln 2$$
$$\frac{\ln 21 - \ln(\ln 2)}{\ln 2} = x$$
$$4.921 = x$$
$$y = 2^{4.921} - 3 = 27.297$$

So the point is (4.921, 27.297)

Derivative of ln *x*:

To find the derivative of $y = \ln x$, we will first rewrite as an exponential and use implicit differentiation:

$$y = \ln x$$
$$e^y = x$$
$$\frac{d}{dx}(e^y) = \frac{d}{dx}x$$
$$e^y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{e^y}$$
$$\frac{dy}{dx} = \frac{1}{x}$$

Using the Inverse Function Th:

$$f(x) = e^z$$
$$f^{-1}(x) = g(x) = \ln x$$
$$g'(x) = \frac{1}{f'(g(x))}$$
$$g'(x) = \frac{1}{e^{\ln x}}$$
$$g'(x) = (f^{-1})'(x) = \frac{1}{x}$$

Theorem: $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$ or $\frac{u'}{u}$

Ex. Find the equation of the line tangent to $y = \ln x$ at $x = 4$

Ex Find the equation of the line passing through the origin and tangent to $y = \ln x$.

Derivative of $\log_a x$:

From an early section: $\log_a x = \dfrac{\ln x}{\ln a}$

To find the derivative:

$$\frac{d}{dx}\log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$$
$$= \frac{1}{\ln a}\cdot\frac{d}{dx}\ln x$$
$$= \frac{1}{\ln a}\cdot\frac{1}{x}$$
$$= \frac{1}{x\ln a}$$

which leads to:

$$\boxed{\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\cdot\frac{du}{dx}}$$

Ex. Find *y'* if $y = \log_a a^{\sin x}$

Logarithmic Differentiation: Sometimes you can use the properties of logs to help simplify before find the derivative.

Ex. Find *y'* if $y = x^x$, $x > 0$

You can not use any of the rules/properties given because of the *x*'s in both the base and the exponent. In this case, we will use logs (natural logs):

$$\ln y = \ln x^x$$
$$\ln y = x\ln x$$
$$\frac{d}{dx}\ln y = \frac{d}{dx}(x\ln x)$$
$$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x(1)$$
$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$
$$\frac{dy}{dx} = y(1+\ln x)$$
$$\frac{dy}{dx} = x^x(1+\ln x)$$

Homework: Pg. 170-171 #3-39 (3's), 41, 43, 44, 47