Chapter 7 – Applications of the Definite Integral

7.1 Integrals as Net Change

Linear Motion

Recall: The velocity is the derivative of the position function.

$$v(t) = s'(t)$$

Therefore: $s(t) = \int v(t)dt$

- Suppose $v(t)$ represents the velocity function of a moving particle. The net distance traveled by the particle from time $t = a$ and $t = b$ is given by:

$$s(t) = \int_a^b v(t)dt$$

Ex. Suppose a particle moving along a horizontal x-axis has a velocity function of

$$v(t) = t^2 - \frac{8}{(t+1)^2}$$

For the time interval $0 \le t \le 5$,

(a) Describe the motion of the particle

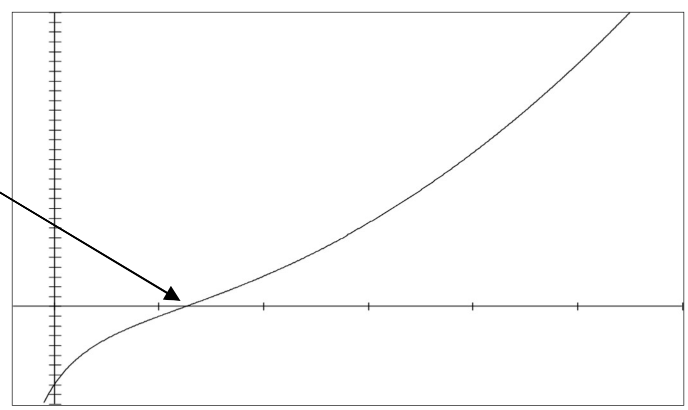
(b) Find the particle's position at

(1) $t = 1$ sec

(2) $t = 5$ sec

(a) The graph of $v(t)$ is shown to the right.

- At t=0, v<0, therefore the particle is initially moving to the left.
- v crosses the x-axis at 1.25454 (using the graphing calculator.
  - So the particle is moving left for $0<t<1.25454$
- Since v is above the x-axis , the particle is moving to the right for $t > 1.25454$



[0,5] by [-10,30]

(b) $s(t) = \int v(t)dt$

(1) for $t = 1$

$$s(t) = \int_0^1 t^2 - \tfrac{8}{(t+1)^2}\,dt$$

$$s(t) = \frac{t^3}{3} + \frac{8}{t+1}\Big|_0^1 = (\tfrac{1}{3} + 4) - (0 + 8) = -3\tfrac{2}{3}$$

$-3\tfrac{2}{3}$ units left of start

(2) for $t = 5$

$$s(t) = \frac{t^3}{3} + \frac{8}{t+1}\Big|_0^5 = (\tfrac{125}{3} + \tfrac{8}{6}) - (0 + 8) = 35$$

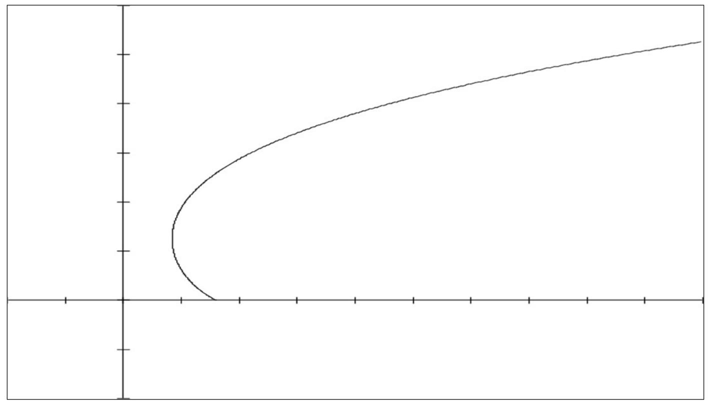
35 units right of the start

Note:
Notice that you don't need to know the starting location of the particle to answer the question.

The answer is the net location from where the starting point is.

If they give you starting point, then add the displacement value to the initial position for the exact location.

- If we were to graph the position function of the particle, then the graph would look like:



  - This shows that path the particle takes.

- Remember $s(t)$ represents the position after time t. This is a "net-change." To find the total distance traveled, we would need to do one of two things:
  1. Take the integral of the portion of the curve that is above the x-axis and subtract the integral of the portion of the curve that is below the x-axis.
  2. Find the integral of the absolute value of the velocity:

$$s_{Total}(t) = \int_a^b |v(t)|dt$$

- You can use your calculators integral function (fnInt) to find this integral.

Ex. A car is moving with initial velocity of 5 mph accelerates at the rate of $a(t) = 2.4t$ mph per sec for 8 sec.

(a) How fast is the car going when the 8 seconds are up?
(b) How far did the car travel during those 8 seconds?

(a) $v_0 = 5$

$$v(t) = \int a(t)dt$$
$$= \int 2.4tdt$$
$$v(t) = 1.2t^2 + C$$

At $t = 0$: $5 = 1.2(0)^2 + C \Rightarrow C = 5$

$$v(t) = 1.2t^2 + 5$$
$$v(8) = 1.2(8)^2 + 5 = 1.2(64) + 5 = 81.8$$

81.8 mph

(b)

$$s(t) = \int |v(t)| dt$$
$$= \int |1.2t^2 + 5| dt$$
$$= 244.8$$

Found using fnInt

244.8 miles

Recall:
(a) A particle is moving to the left if $v(t) < 0$
(b) A particle is moving to the right if $v(t) > 0$
(c) A particle is "stopped" if $v(t) = 0$

Ex. A particle is moving along a straight line with a velocity $v(t) = \cos \pi \cdot t$. If $s(0)=0$,

(a) Describe the particles motion
(b) What position is the particle at $t = 3$?
(c) What is the total distance the particle has moved in the first 3 seconds?

(a) Look for when the particle stops

$$v(t) = \cos(\pi \cdot t) = 0$$
$$\pi \cdot t = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{5\pi}{2}, \ldots$$
$$t = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots$$

Using the first derivative test

$$\begin{array}{ccccccccccc} & + & & - & & + & & - & & + & \\ 0 & & \tfrac{1}{2} & & \tfrac{3}{2} & & \tfrac{5}{2} & & \tfrac{7}{2} & & \tfrac{9}{2} \end{array}$$

So the particle is moving:
Right: $0 < t < ½$, $\tfrac{3}{2} < t < \tfrac{5}{2}, \ldots$
Left: $\tfrac{1}{2} < t < \tfrac{3}{2}$, $\tfrac{5}{2} < t < \tfrac{7}{2}$, …

(b)

$$s(t) = \int v(t)dt$$
$$= \int \cos \pi \cdot tdt$$
$$s(t) = \tfrac{1}{\pi} \sin \pi \cdot t + C$$

At $t = 0$: $s(0) = 0 \Rightarrow C = 0$

$$s(3) = \tfrac{1}{\pi} \sin(3\pi) = \tfrac{1}{\pi} \cdot 0 = 0$$

So it is back at the starting point.

(c)

$$s_{total}(t) = \int |v(t)| dt$$
$$= \int |\cos \pi \cdot t| dt$$
$$= 1.90974$$

Ex. A particle is moving along a straight line with a velocity $v(t) = 3t^2 - 8t + 4$. If $s(0)=5$, then

(a) Describe the particle's motion
(b) What position is the particle at $t = 4$?
(c) What is the total distance the particle has moved in the first 4 seconds?

(a) Look for when the particle stops

$$v(t) = 3t^2 - 8t + 4$$
$$0 = (3t - 2)(t - 2)$$
$$t = \tfrac{2}{3}, 2$$

First derivative test:

$$\begin{array}{ccccccc} & -\,- & & +\,- & & +\,+ & \\ 0 & + & \tfrac{2}{3} & - & 2 & + & \end{array}$$

Right left right

Moving left: $\tfrac{2}{3} < t < 2$
Moving right: $0 < t < \tfrac{2}{3}$ or $t > 2$

(b)

$$s(t) = \int v(t)dt$$
$$= \int 3t^2 - 8t + 4dt$$
$$s(t) = t^3 - 4t^2 + 4t + C$$
$$5 = 0^3 - 4 \cdot 0^2 + 4(0) + C$$
$$5 = C$$

$$s(t) = t^3 - 4t^2 + 4t + 5$$
$$s(4) = 4^3 - 4(4)^2 + 4(4) + 5 = 21$$

(c)

$$s_{total}(t) = \int |v(t)| dt$$
$$= \int |3t^2 - 8t + 4| dt$$
$$= 18.370$$

Homework: Pg 386 # 1 – 7 odd, 9 – 16

7.2 Areas in the Plane

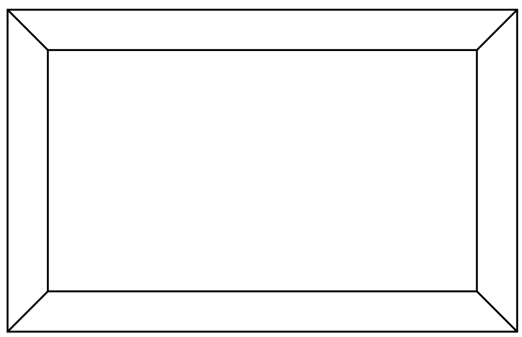
Area Between 2 Curves

- In previous chapters we found the area under a curve, $y = f(x)$, on the interval $a \le x \le b$ is the integral:

$$\int_a^b f(x)dx$$

- So how do you find the area between 2 curves?
    - Ok, let's look at the following problem:

Ex. A painting that is 36 in by 24 in has a 2 in frame around the outside of it. What is the area of the frame?



$$\begin{aligned} A_F &= A_{P\&F} - A_P \\ &= 40(28) - 36(24) \\ &= 256\,in^2 \end{aligned}$$

- This process works the same for finding the area between 2 curves.

Def: ***Area Between Curves***

If *f* and *g* are continuous with $f(x) \le g(x)$ throughout the interval [*a, b*], then the **area between the curves *y* = *f*(*x*) and = *g*(*x*) from *a* to *b*** is the integral [$f - g$] from *a* to *b*.

$$A = \int_a^b [f(x) - g(x)]dx$$

Ex. Find the area of the region bounded by $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \frac{\pi}{4}$

$$A = \int_a^b [f(x) - g(x)]dx = \int_0^{\pi/4} [\sec^2 x - \sin x]dx = [\tan x + \cos x]_0^{\pi/4}$$

$$= \left(\tan\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - (\tan 0 + \cos 0)$$

$$= \left(1 + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \frac{\sqrt{2}}{2}$$

- If the bounds of integration are not given to you, then use the points of intersection of the two curves as the bounds.

Ex. Find the area of the region enclosed by $y = 2 - x^2$ and $y = -x$.

Steps

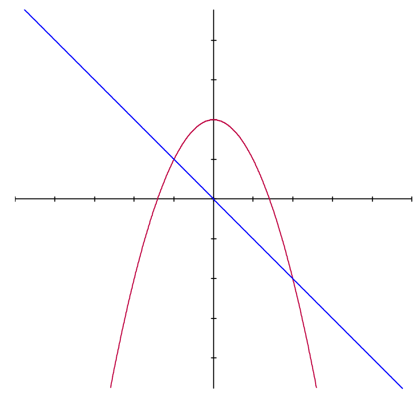
1. Graph the functions
2. If no bounds, find the points of intersection.
3. Set up the integral
4. Integrate and evaluate.

$$A = \int_{-1}^{2} (2 - x^2 - (x))dx$$

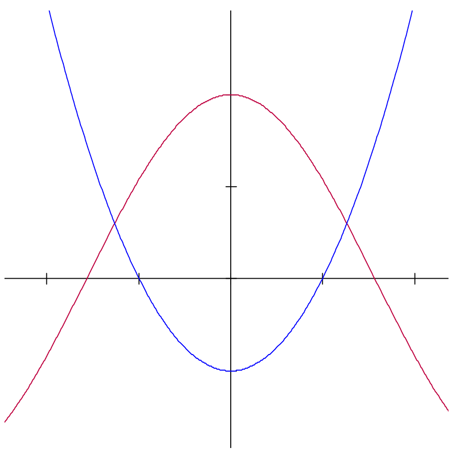
$$= \int_{-1}^{2} 2 - x^2 + xdx$$

$$= \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^{2} = [4 - \tfrac{8}{3} + 2] - [-2 + \tfrac{1}{3} + 1/2]$$

$$= 6 - \tfrac{8}{3} + 2 - \tfrac{1}{3} - 1/2 = 8 - 3 - 1/2 = 4\tfrac{1}{2}$$



Ex. Find the area of the region enclosed by the graphs of $y = 2\cos x$ and $y = x^2 - 1$

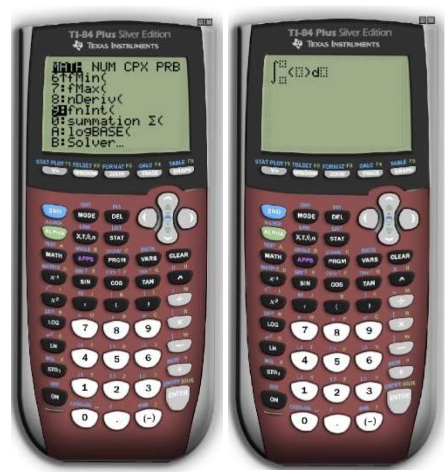


So cos is on top: Have to use grapher to find the points of intersection.
x= -1.265423 and x = 1.265423

$$A = \int_{-1.265423}^{1.265423} 2\cos x - (x^2 - 1)dx$$

This has to be done as well on your calculator:

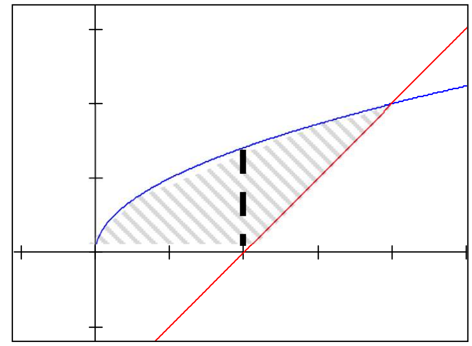
$$fnInt(2\cos x - (x^2 - 1), x, -1.265423, 1.265423)$$



- There are times when you can't simply use $[f - g]$ for your integral.

- If a boundary of a region is defined by more than one function, we partition the region into 2 or more sub-regions that correspond to the function.

Ex. Find the area of the region ***R*** in the first quadrant that is bounded by $y=\sqrt{x}$, $y=x-2$, and the x-axis.



You can set it up 3 different ways but you need to find the point of intersection.

$$\sqrt{x}=x-2$$
$$x=(x-2)^2$$
$$x=x^2-4x+4$$
$$0=x^2-5x+4$$
$$0=(x-4)(x-1)$$

x = 4, 1 so we will use x = 4

Where does the line cross x-axis?

$$x-2=0$$
$$x=2$$

Method 1:

$$\int_0^4 \sqrt{x}dx-\int_2^4 (x-2)dx$$

Method 2: Partition the area

$$\int_0^2 \sqrt{x}dx+\int_2^4 \sqrt{x}-(x-2)dx$$

Method 3: Use area between curves first

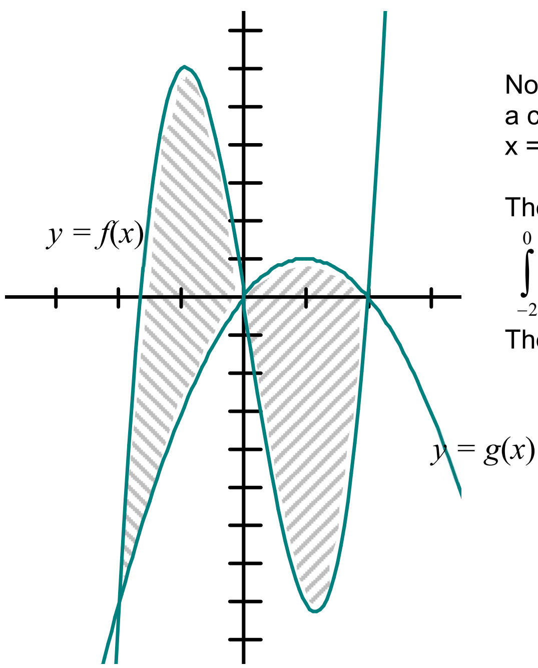
$$\int_0^4 \sqrt{x}-(x-2)dx-\left[-\int_0^2 x-2dx\right]$$

All three of them result in the same value: $\frac{10}{3}=3.\overline{3}$

Calculator: 3.3333333544

- What if more than one region is created?

Ex. Find the area of the region bounded by $f(x)=3x^3-x^2-10x$ and $g(x)=-x^2+2x$



Note the two regions formed. You need to find the 3 points of intersection (use a calculator)
x = –2, 0, 2

There are two regions, so find areas of each and add:

$$\int_{-2}^{0}(3x^3-x^2-10x)-(-x^2+2x)dx+\int_0^2(-x^2+2x)-(3x^3-x^2-10x)dx$$

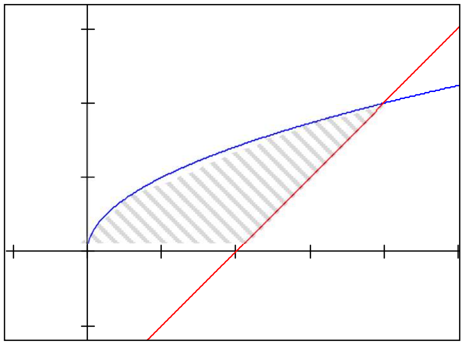
The result using the calculator is 24.

- Suppose you only integrate the previous example from –2 to 2?

You get 0

- Sometimes the boundaries of a region are more easily described by functions of y than with functions of x.
  - Instead of areas of vertical rectangles, we use horizontal rectangles.

Ex. Find the area of the region ***R*** in the first quadrant that is bounded by $y=\sqrt{x}$, $y = x-2$, and the x-axis by integrating with respect to y.



If you look at the region as you move along the y-axis, you see it is a simple region of an area between 2 curves.

So…solve each equation of x in terms of y and set up the integral (instead of higher-lower use rightmost-leftmost)

$y=\sqrt{x}$

$y^2 = x$

$y = x-2$

$y+2=x$

Get the y-coordinate of the point of intersection:

$y^2 = y+2$

$y^2 - y - 2 = 0$

$(y-2)(y+1)=0$

y = 2, –1 so use y = 2

$$\int_0^2 y+2-y^2 dy = 3.\overline{3}$$

.

- You can always make your own decision on whether to integrate with respect to x or y.

You can also utilize Geometry in finding the area of this example.

In this case, you find the area under the curve and then subtract off the area of the triangle under it.

$$\text{Area} = \int_0^4 \sqrt{x}\, dx - \frac{1}{2}(2)(2) = \frac{2}{3}x^{3/2}\Big]_0^4 - 2 = \frac{10}{3}$$

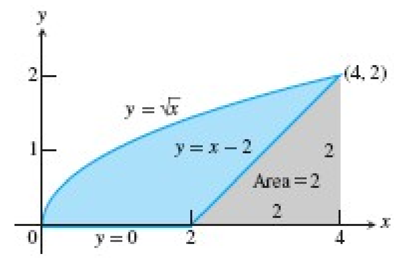
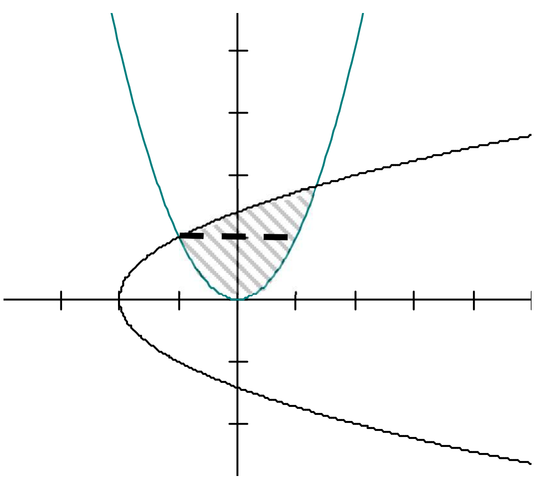


Figure 7.14 The area of the blue region is the area under the parabola $y=\sqrt{x}$ minus the area of the triangle. (Example 7)

Ex. Find the area of the region bounded by the graphs of $y=x^2$ and $x = y^2 - 2$



It is best to go along the x-axis since it is a simple curve over curve. Going along the y-axis puts the curve over itself (not a good choice) and you would need to partition the region.

Hard part:

Points of Intersection – use calculator but you need to put in the correct formulas: $y=x^2$ and $y=\sqrt{x+2}$ (solving for y - use the positive square root since it is the upper piece you are using.

Using the graphing calculator you get x = –1, 1.35321

$$\int_{-1}^{1.35321} (\sqrt{x+2} - x^2)dx = 2.267558691 \approx 2.268$$

If you chose to work along the y-axis, you would use the following integral: (partition using the dashed line)

$$\int_0^1 \sqrt{y} - (-\sqrt{y})dy + \int_1^{1.8311773} \sqrt{y} - (y^2-2)\ dy \approx 2.268$$

The 1.8311773 in the upper bound is found using $y=x^2$ at x = 1.35321

Homework:
Day 1: Pg. 380 – 381 #1-11, 13, 15, 31, 33

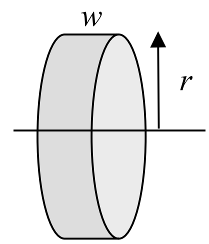
Day 2: Pg. 381 #12-18even, 19-29, 32, 34

7.3 Volumes

Volume as an Integral

- Another application of the integral is finding the volume of a three-dimensional solid.
- The first type of solid is the Solid of Revolution
  - Examples are axles, funnels, pills, bottles, and pistons
  - These solids are formed by taking a two-dimensional region and revolving it around a line, called the *axis of revolution*.
  - The simplest of these solids is the right circular cylinder or ***disc***.

The Disc Method

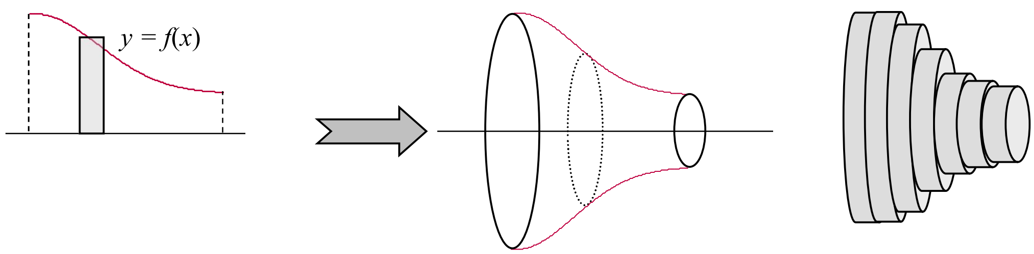


- The volume of a disc (or right circular cylinder) is $V = \pi r^2 h$
- A better way of looking at it is

$$V = (\text{area of the disc})*(\text{width of the disc})$$
$$= \pi r^2 w$$

where r is the radius of the disc and w is the width of the disc.

- Suppose you have a curve $y = f(x)$ revolved around the axes



- As you can see, a series of discs of equal width will "create" the solid. To determine the volume of this solid, use the volume of each disc:

$$\Delta V = \pi r^2 \Delta x$$

where Δx is the width of the disc. r is the radius of the current disc, ΔV is the volume of the current disc.

- Adding all of the discs created, we get:

$$V_{solid} = \sum_{i=1}^{n} \pi [R(x_i)]^2 \Delta x$$
$$= \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x$$

where $R(x_i)$ is the radius of disc i

- The smaller you make the width of the of disc, the closer this estimate approaches the actual volume. So….

$$V = \lim_{\Delta x \to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x$$
$$= \pi \int_a^b [R(x)]^2 \, dx$$

- Which leads to the following

To find the volume of a solid of revolution with the *disc method*, use one of the following:

Over x-axis: $V = \pi \int_a^b [R(x)]^2 \, dx$ where a and b are the upper and lower bounds

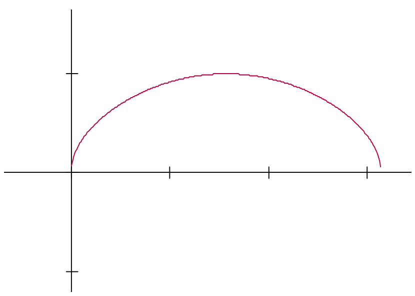
Over y-axis: $V = \pi \int_c^d [R(y)]^2 \, dy$ where c and d are the upper and lower bounds

- Applying this one step further:

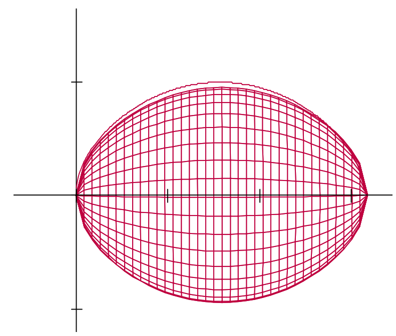
Let $y = f(x)$ be a function. The volume of the solid of revolution of $y=f(x)$ or $x=g(y)$ revolved around:

1. the x-axis on the interval $[a,b]$ is $V = \pi \int_a^b [f(x)]^2 \, dx$
2. the y-axis on the interval $[c,d]$ is $V = \pi \int_c^d [g(y)]^2 \, dy$

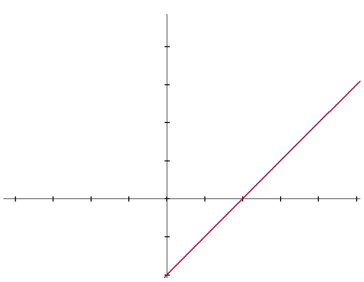
ex. Find the volume of the solid formed by $f(x) = \sqrt{\sin x}$ and the x-axis from $[0, \pi]$ revolved around x-axis.



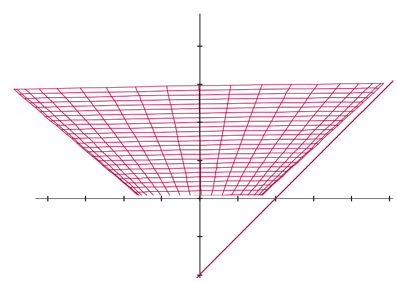
$$V = \pi \int_0^{\pi} \left(\sqrt{\sin x}\right)^2 dx$$
$$V = \pi \int_0^{\pi} \sin x \, dx$$
$$V = \pi[-\cos x]_0^{\pi}$$
$$V = \pi\big(-(-1) - (-1)\big)$$
$$V = 2\pi$$



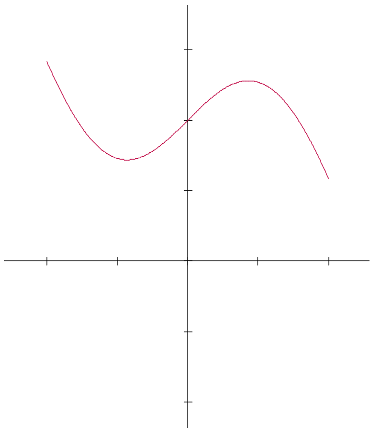
Ex. Find the volume of the solid formed by $f(x) = x - 2$, y-axis, on the interval $0 \le y \le 3$ revolved around the y-axis.



$$V = \pi \int_0^3 (y+2)^2 \, dy$$
$$= \pi \left[\frac{(y+2)^3}{3} \right]_0^3$$
$$= \pi \left[\frac{5^3}{3} - \frac{2^3}{3} \right]$$
$$= \pi \left[\frac{125-8}{3} \right] = \frac{117\pi}{3} = 39\pi$$

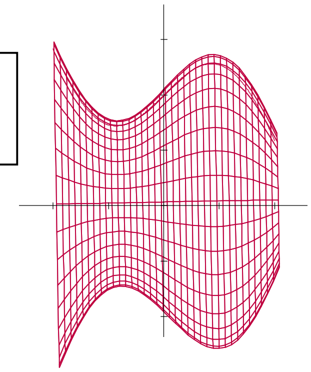


Ex. The region between the graph of $f(x) = 2 + x\cos x$ and the x-axis over the interval [–2,2] is revolved about the x–axis to generate a solid. Find its volume.



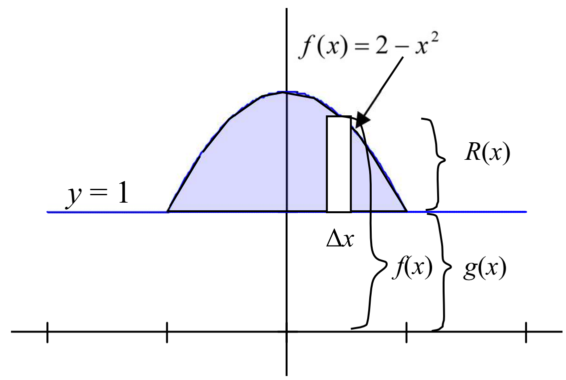
$$V = \pi\int_{-2}^{2}(2 + x\cos x)dx$$
$$= 55.85630631$$
$$= 55.856$$

Use calculator
Or By Parts!!



Suppose the axis of revolution IS NOT a coordinate axis….

Ex. Find the volume of the solid formed by revolving the region bounded by $f(x) = 2 - x^2$ and $g(x) = 1$ about the line $y = 1$. (see graph)



Setting the two functions equal to each and solving, you find the points of intersection @ $x = –1$ and $x = 1$.

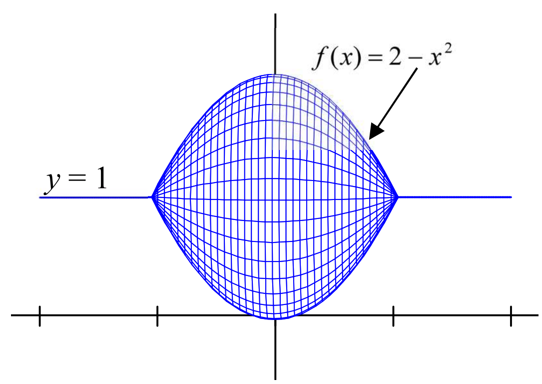
In order to use the disc method, you need to know the radius of each cross section. To find the radius:

$$R(x) = f(x) - g(x)$$

The outside radius-inside radius (from the parallel axis)
Therefore:

$$R(x) = 2 - x^2 - 1$$
$$R(x) = 1 - x^2$$

Now we integrate:



$$V = \pi\int_a^b [R(x)]^2\,dx$$
$$= \pi\int_{-1}^{1}(1 - x^2)^2\,dx$$
$$= \pi\int_{-1}^{1}(1 - 2x^2 + x^4)dx = \pi\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1} = \frac{16\pi}{15}$$

So, if you are revolving around a line that is not an axis, you calculate the radius function:

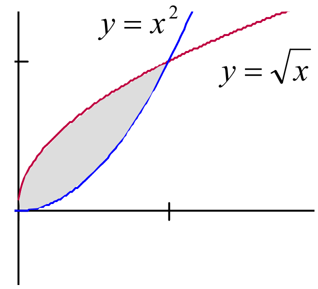
1. Parallel to x-axis: $R(x) = f(x) - g(x)$ where $f(x)$ is the function further from x-axis
2. Parallel to y-axis: $R(y) = f(y) - g(y)$ where $f(y)$ is the function further from y-axis

Homework: Pg 392-393 #13–20, 28, 29, 30

7.3 Volumes (Day 2)

Suppose the solid is not bounded by the axis of revolution…

Ex. Find the volume of the solid formed by revolving the region bounded by the graphs of $y=\sqrt{x}$ and $y=x^2$ about the x-axis.



Looking at the graphs to the right, you can see after revolving it, we end up with an "open bowl" with the darker section being the section "removed" from the solid.

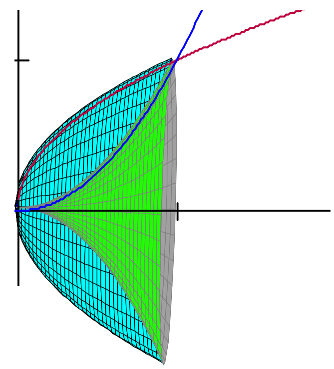
Therefore it not a complete solid, rather a partial solid. Instead of using a solid disc to determine the volume, we will use a ***WASHER METHOD.***

The volume of any washer whose outer radius is R and inner radius is r is given by:

$$V=\pi R^2 w-\pi r^2 w \quad \Longrightarrow \quad V=\pi(R^2-r^2)w$$

Therefore the corresponding volume of revolution formula would be:

$$V=\pi\int_a^b([R(x)]^2-[r(x)]^2)dx$$



Solution:
1. Find the bounds a and b: set two equations equal and solve
2. Plug into fomula and evaluate.

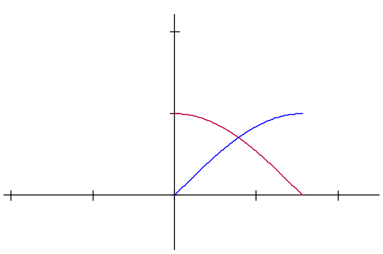
$$\sqrt{x}=x^2$$
$$x=x^4$$
$$0=x^4-x$$
$$0=x(x^3-1)$$

$x=0$	$x^3-1=0$
	$x^3=1$
	$x=1$

So…

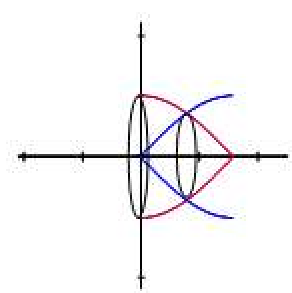
$$V=\pi\int_0^1(\sqrt{x})^2-(x^2)^2dx$$
$$=\pi\int_0^1(x-x^4)dx$$
$$=\pi\left[\tfrac{1}{2}x^2-\tfrac{1}{5}x^5\right]_0^1=\pi[(\tfrac{1}{2}-\tfrac{1}{5})-(0-0)]=\pi(\tfrac{3}{10})=\tfrac{3\pi}{10}$$

ex. The region in the first quadrant enclosed by the y-axis and the graphs of $y=\cos x$ and $y=\sin x$ is revolved about the x-axis to form a solid. Find its volume.



The two curves intersect at $\pi/4$. Cos is higher than sin…….

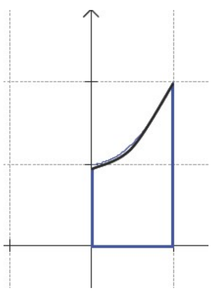
$$V=\pi\int_0^{\pi/4}\cos^2 x-\sin^2 x\,dx$$
$$=\pi\int_0^{\pi/4}\cos 2x\,dx$$
$$=\pi\left[\tfrac{1}{2}\sin 2x\right]_0^{\pi/4}$$
$$=\frac{1}{2}\pi\left[\sin\frac{\pi}{2}-\sin 0\right]$$
$$=\frac{1}{2}\pi(1-0)=\frac{1}{2}\pi$$



At times, you may need to use two-integrals to do the work…

Ex. Find the volume of the solid formed by revolving the region bounded by the graphs $y = x^2 + 1$, y=0, x=0, and x=1 about the y-axis.

Since revolving around y, we need the equation in terms of x.
Notice from y = 0 to y = 1, it is a solid cylinder, and from 1 to 2, it is hollowed out (washer!)
Combine two integrals: $V = V_{lower} + V_{upper}$



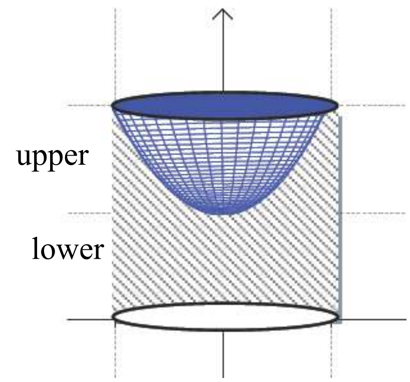
$$V = \pi \int_0^1 R^2\,dy + \pi \int_1^2 R^2 - r^2\,dy$$

$$V = \pi\left[\int_0^1 1^2\,dy + \int_1^2 1^2 - \left(\sqrt{y-1}\right)^2 dy\right]$$

$$V = \pi\left[\int_0^1 1^2\,dy + \int_1^2 1^2 - (y-1)\,dy\right]$$

$$V = \pi\left[y\Big|_0^1 + \left(2y - \frac{y^2}{2}\right)\Big|_1^2\right]$$

$$V = \pi\left[(1-0) + \left\{((4-2) - (2-\tfrac{1}{2})\right\}\right] = \pi\left[1+\frac{1}{2}\right] = \frac{3}{2}\pi$$



<u>Solids with Known Cross Sections</u>

- The disc method can be generalized to solids of any shape.
  - provided you know a formula for the area of an arbitrary cross section.

<u>Def</u>: Volume of Any Solid

The volume of any solid of known integrable cross section area $A(x)$ from $x=a$ to $x=b$ is the integral of A from a to b:

1. Cross sections of area $A(x)$ taken perpendicular to the x-axis: $V = \int_a^b A(x)dx$
2. Cross sections of area $A(y)$ taken perpendicular to the y-axis: $V = \int_c^d A(y)dy$

ex. A pyramid of 3 m high has congruent triangular sides and a square base that is 3 m on each side. Each cross section of the pyramid parallel to the base is a square. Find the volume of the pyramid.

$$A = b^2$$

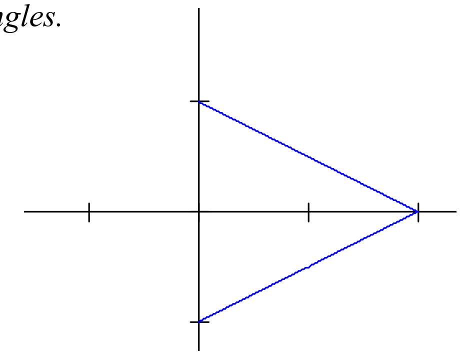
$$A(x) = x^2$$

$$V = \int_a^b A(x)dx = \int_0^3 x^2 dx = \left[\frac{x^3}{3}\right]_0^3 = \left[\frac{27}{3} - \frac{0}{3}\right] = 9$$

$$V_{pyr} = \frac{1}{3}Bh = \frac{1}{3}(3)^2(3) = 9$$

Ex. Find the volume of the solid shown in the diagram to the right. The base of the solid is the solid is the region bounded by the lines $f(x)=1-\frac{x}{2}$, $g(x)=-1+\frac{x}{2}$, and $x=0$.

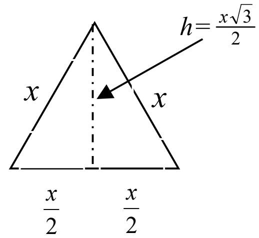
The cross sections perpendicular to the x-axis are equilateral triangles.



The base and area of each triangular cross section are:

(The base is always the difference between the 2 functions!)

Base $= (1-\frac{x}{2})-(-1-\frac{x}{2}) = 2-x$



Area $= \frac{\sqrt{3}}{4}(base)^2$

$A(x) = \frac{\sqrt{3}}{4}(2-x)^2$

This is the function using the length of the base you found earlier

This is always the base formula (the one you learned in Geometry!

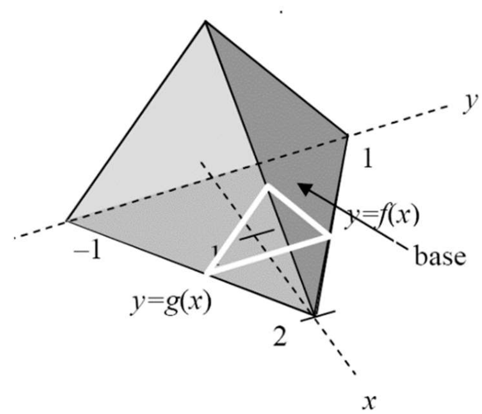
Therefore:

$$V=\int_a^b A(x)dx$$

$$=\int_0^2 \frac{\sqrt{3}}{4}(2-x)^2\,dx$$

$$=-\frac{\sqrt{3}}{4}\left[\frac{(2-x)^3}{3}\right]_0^2 = -\frac{\sqrt{3}}{4}\left[\frac{(2-2)^3-(2-0)^3}{3}\right]$$

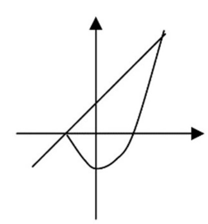
$$=-\frac{\sqrt{3}}{4}\left[\frac{0-8}{3}\right]=\frac{8\sqrt{3}}{12}=\frac{2\sqrt{3}}{3}$$



ex. Find the volume of the solid whose base is bounded by the graphs of $y=x+1$ and $y=x^2-1$, with the indicated cross sections taken perpendicular to the x-axis

(a) Squares

(b) Rectangles of height 1



Basic Steps:
1. Find length of base (top-bottom)
2. Write the appropriate area formula for the CROSS SECTION
3. Substitute the length of base expression into the area formula
4. Integrate on the bounds given or determined

(a)

Base = Square Area $= s^2$

Side $= x+1-(x^2-1) = 2+x-x^2$

$A(x) = (2+x-x^2)^2$

$V=\int_{-1}^{2} A(x)dx$

$V=\int_{-1}^{2} 4+4x-3x^2-2x^3+x^4\,dx$

$V = 4x+2x^2-x^3-\frac{1}{2}x^4+\frac{1}{5}x^5\Big|_{-1}^{2}$

$V=\left(8+8-8-8+\frac{32}{5}\right)-(-4+2+1-\frac{1}{2}-\frac{1}{5})$

$V=\frac{32}{5}+\frac{3}{2}+\frac{1}{5}=\frac{81}{10}$

(b)

Area $= bh = b(1) = b$

$b = 2+x-x^2$

Type equation here.

$A(x) = 2+x-x^2$

$V=\int_{-1}^{2} 2+x-x^2\,dx$

$V = 2x+\frac{1}{2}x^2-\frac{1}{3}x^3\Big|_{-1}^{2}$

$V=\left(4+2-\frac{8}{3}\right)-\left(-2+\frac{1}{2}+\frac{1}{3}\right)$

$V = 8-\frac{1}{2}-3 = \frac{9}{2}$

Bounds: $x+1=x^2-1$

$x^2-x-2=0$

$(x-2)(x+1)=0$

$x=2 \quad x=-1$

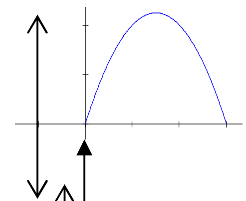
Homework: Pg. 392-393 #21-27, 31-34

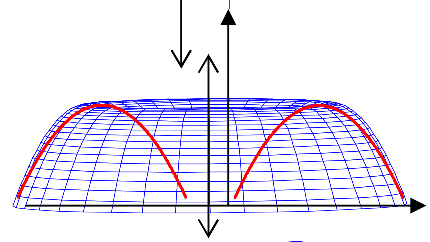
7.3 Volumes (Day 3)

Cylindrical Shells:

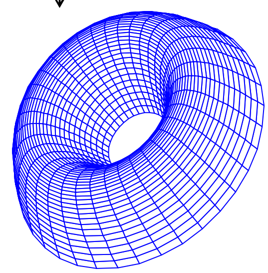
- There is another way to find volumes of solids of rotation that can be useful when the axis of revolution is PERPENDICULAR to the axis containing the interval of integration.
  - In other words: revolving around y-axis, but interval is along x-axis.

Ex. The region enclosed by the x-axis and the parabola $y=3x-x^2$ is revolved about the line $x=-1$ to generate the shape of a Bundt cake. What is the volume of the cake?

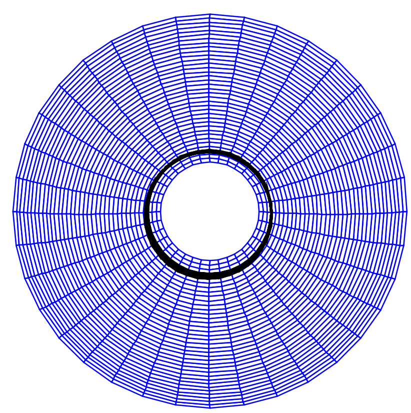




- The top graph to the right is the curve and the line to rotate about
- The graph below is the solid after revolving about the line $x=-1$
- The bottom figure is a more top view of the bundt cake



- Now since the curve is being rotated about a line parallel to the y-axis, we would normally use the y-axis form of the disc method. But the curve region is not easily manipulated if solved for x. Therefore, we will use the Shell Method which integrates with respect to x not y.
  - The cross section will be cut in an unusual way:
    - Instead of cutting the usual wedge shape, cut a cylindrical slice by cutting straight down all the way around close to the inside hole (basically making a cylinder).
      - See the bottom diagram. The darker line show the cylindrical shell from the top.
  - Cut another shell outside of that, etc….
  - The radii of the cylinders gradually increase and the height follows the shape of the parabola (increasing then decreasing).
    - The radius is always $1+x_k$
    - The height is the $3x_k - x_k^2$
  - Unrolling the cylindrical shell and laying it flat would make it a rectangle



    - The height of the rectangle is the $3x_k - x_k^2$
    - The width of the rectangle is the circumference of the circle, $2\pi(3x_k - x_k^2)$
    - The thickness of the shell is Δx
    - Therefore the volume of the shell is:

$$V = 2\pi(x_k+1)(3x_k - x_k^2)\Delta x$$

    - The sum of all these shells would be:

$$\sum_{k=1}^{n} 2\pi(x_k+1)(3x_k - x_k^2)\Delta x$$

    - Which by definition is: $2\pi\int_0^3 (x+1)(3x-x^2)dx$
  - This leads to the following:

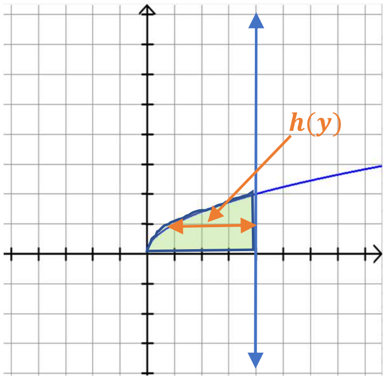
To find the volume of a solid of revolution with the shell method, use one of the following:

Horizontal Axis of Revolution: $V = 2\pi\int_c^d p(y)h(y)dy$

Vertical Axis of Revolution: $V = 2\pi\int_a^b p(x)h(x)dx$

p = shell radius h = shell height In many cases, $p(x) = x$ or $p(y) = y$ (about an axis)

Ex. Find the volume of the solid formed by revolving the region bounded by the curve $y=\sqrt{x}$, the x-axis, and the line $x = 4$ about the x-axis.



Horizontal: need to use the y interval [0,2]
the shell height is the right most distance – left most
$h(y) = 4 - y^2$

$$V = 2\pi \int_a^b y\, h(y)\, dy = 2\pi \int_0^2 y(4-y^2)\, dy$$

$$V = 2\pi \int_0^2 4y - y^3\, dy = 2\pi \left[4y - \frac{y^4}{4}\right]_0^2$$

$$V = 2\pi \left[\left(8 - \tfrac{16}{4}\right) - (0-0)\right] = 2\pi\,(4) = 8\pi$$

By Disc Method:

$$V = \pi \int_0^4 \left(\sqrt{x}\right)^2 dx$$

$$V = \pi \int_0^4 x\, dx = \pi \left[\frac{x^2}{2}\right]_0^4$$

$$V = \pi \left[\frac{16}{2} - 0\right] = 8\pi$$

There are times that the Disc Method is better than Shells, but either one will work!!!

As long as the cross sections of the solid have areas that can be described using some formula, you can find the volume.

- Drawing is not too easy so bear with it.

Ex. A mathematician has a paperweight made so that its base is the shape of the region between the x-axis and one arch of the curve $y = 2\sin x$ (in linear units in inches). Each cross section cut perpendicular to the x-axis is a semicircle whose diameter runs from the x-axis to the curve. Find the volume of the paperweight.

This is a Known Cross Section problem (see last section).

Base – semi circle: Area $= \frac{1}{2}\,\pi\, r^2$

Radius is half the diameter (the height of the curve): $\frac{1}{2}\,(2\sin x) = \sin x$

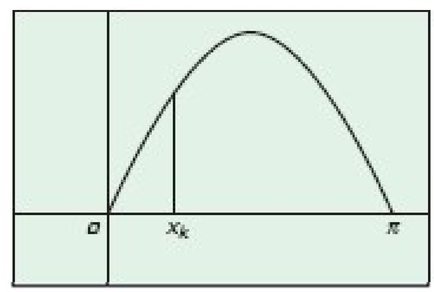
So the cross-section area formula is:

$$A = \frac{1}{2}\,\pi(\sin x)^2$$

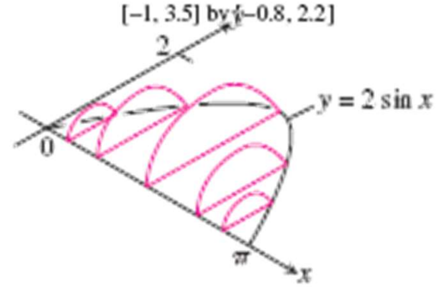
So the volume is $V = \int_a^b A(x)\, dx$

$$V = \frac{\pi}{2}\int_0^\pi (\sin x)^2 dx = \frac{\pi}{2}\int_0^\pi \frac{1}{2}(1-\cos 2x)\, dx$$

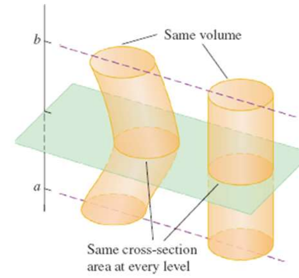
$$V = \frac{\pi}{4}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi = \frac{\pi}{4}[(\pi-0)-(0-0)] = \frac{\pi^2}{4}\ \text{in}^3$$



[−1, 3.5] by [−0.8, 2.2]



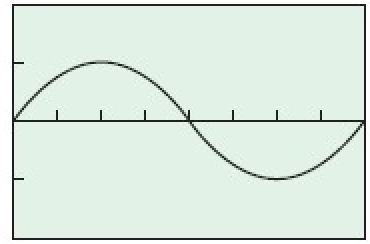
<u>Cavalieri's Volume Theorem</u>:
If two solids have the same altitudes and identical cross section areas at EACH height will have the same volume.



Homework: Pg. 393-394 #35-42, 45, 49

7.4 Lengths of Curves

Question: How long is one cycle of the sine wave $y = \sin x$?



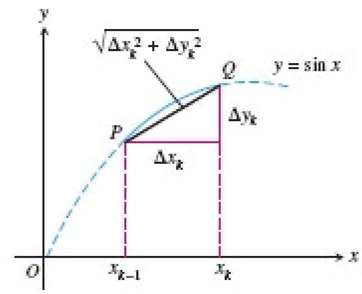
[0, 2π] by [−2, 2]

Ex. What is the length of the curve $y = \sin x$ from $x = 0$ to $x = 2\pi$?

To find the length, we will use integration. We don't have a formula from Algebra/Geometry/Trig to find the length.

Suppose we look at a section of the curve between two points. If we calculate the distance between the points, and add all the sections together, we can find an approximate the length of the curve.

$$\sum \sqrt{\Delta x^2 + \Delta y^2}$$



The catch is:
We need to use the shortest segment possible. (see figure to the right)
This turns out to be a Riemann Sum. If we divide the curve into an infinite number of intervals, then

$$\sum \sqrt{\Delta x^2 + \Delta y^2} = \sum \frac{\sqrt{{\Delta x_k}^2 + {\Delta y_k}^2}}{\Delta x_k} \Delta x_k$$

Bringing the denominator into the radical (squaring it in the process, we end up with:

$$\sum \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} \; \Delta x_k$$

Turning this into an integral, we get: $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

So back our problem: $L = \int_0^{2\pi} \sqrt{1 + (\cos x)^2}\, dx = 7.640394478 \ldots \approx 7.64$

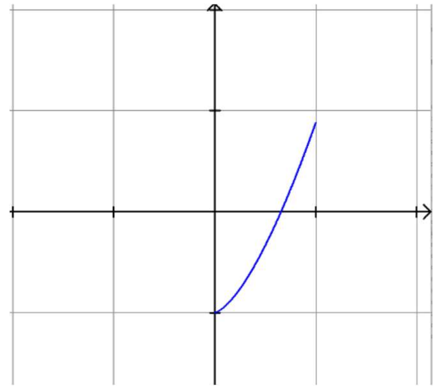
Theorem: The Length of a Smooth Curve

The length of the curve $y = f(x)$ from (a, c) to (b, d) is

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx,$$ *if y is a smooth function on x on* [*a*, *b*]

$$L = \int_c^d \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy,$$ *if x is a smooth function on y on* [*a*, *b*]

Ex. Find the exact length of the curve $y = \frac{4\sqrt{2}}{3}\, x^{3/2} - 1$ for $0 \le x \le 1$



Ok……. $\frac{dy}{dx} = 2\sqrt{2}\, x^{\frac{1}{2}} = 2\sqrt{2x}$

Checking the graph to the right to see in continuous on the interval: It is

$$L = \int_0^1 \sqrt{1 + \left(2\sqrt{2x}\right)^2}\, dx = \int_0^1 \sqrt{1+8x}\, dx$$

Using u-substitution $u = 1 + 8x$ $\quad du = 8dx \quad \frac{1}{8}du = dx$

(bounds change: $0 \to 1 \quad 1 \to 9$

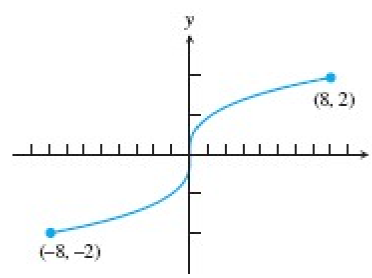
$$L = \frac{1}{8}\int_1^9 u^{\frac{1}{2}}\, du = \frac{1}{8}\left[\frac{2}{3}\, u^{\frac{3}{2}}\right]_1^9 = \frac{1}{12}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{1}{12}[27-1] = \frac{26}{12} = \frac{13}{6} \text{ units}$$

Ex. Find the length of the curve $y = x^{\frac{1}{3}}$ between (–8, –2) and (8, 2).

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$$

<u>ISSUE</u>!!!!! ***Derivative not defined at x = 0.*** **So you can't use the integral with respect to *x*.**

So, use the integral with respect to y and see if that is better…….



$y = x^{\frac{1}{3}} \quad \rightarrow \quad x = y^3$

$\frac{dx}{dy} = 3y^2$

$$L = \int_{-2}^{2} \sqrt{1 + [3y^2]^2}\, dy \approx 17.26 \text{ units}$$

Definitely a calculator problem (for now!!)

So, if you end up with a derivative that is undefined for part of the interval, then try using the other form of the equation.

Homework: Pg. 416 – 417 #1-9odd (use graphing calc), 12, 15, 18, 22, 23, 25, 27, 32-37

7.5a Applications from Science and Statistics (Day 1)

Values to know:
4.4 N = 1 lb.

(1 N) (1 m) = 1 Nm = 1 Joule (J)

Recall: **Work** - $W = Fd$ (force x displacement)

If the force is not a constant value, then the work done moving an object from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx$$

Ex. Find the work done by the force $F(x) = \cos x$ newtons along the x–axis from $x = 0$ meters to $x = ½$ meters.

$$W = \int_0^{\frac{1}{2}} \cos(\pi x)\,dx$$

$$W = \frac{1}{\pi}\sin(\pi x)\Big|_0^{\frac{1}{2}}$$

$$W = \frac{1}{\pi}\left[\sin\tfrac{\pi}{2} - \sin 0\right] = \frac{1}{\pi}(1-0) = \frac{1}{\pi} \approx \ 0.318$$

Ex. A leaky bucket weight 22N empty. It is lifted from the ground at a constant rate to a point 20m above the ground by a rope weighing 0.4N/m. The bucket starts with 70N (approx. 7.1 L) of water, but it leaks at a constant rate and just finishes draining as the bucket reaches the top. Find the amount of work done:

(a) Lifting the bucket alone
(b) Lifting the water alone
(c) Lifting the rope alone
(d) Lifting the bucket, water, and rope together

(a) Constant rate: $W = (22N) \times (20\,m) = 440$ J

(b) Because it leaks steadily, the amount of water decreases. The Force function is:

$$F(x) = 70\left(\frac{20-x}{20}\right) = 70\left(1 - \frac{x}{20}\right) = 70 - 3.5x$$

Original weight   Proportion left at elevation x

$$W = \int_0^{20} (70 - 3.5x)dx = [70x - 1.75x^2]_0^{20} = 1400 - 700 = 700 \text{ J}$$

(c) The rope is not constant as well due to change in length:
Starting: 0.4(20)=8N when on the ground and 0 N at the top. The rate of change is constant: $20 - x$. So, $F(x) = 0.4(20 - x) = 8 - 0.4x$

$$W = \int_0^{20} 8 - 0.4x\,dx = [8x - 0.2x^2]_0^{20} = (160 - 80) - (0 - 0) = 80 \text{ J}$$

(d) $Total\ Work = 440 + 700 + 80 = 1220$ J

Ex. A conical tank shown to the right is filled within 2 ft of the top with olive oil weighing 57 lbs/ft^3. How much work does it take to pump the oil to the rim of the tank?

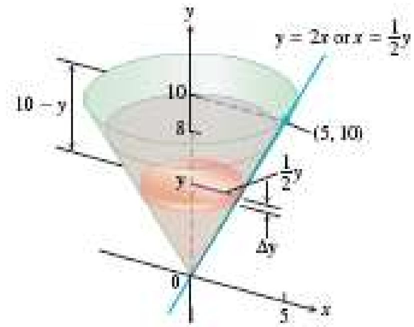
Think of the oil as "coins" of oil stacked one on the other. In the diagram imagine the 8 as the top coin of oil (not the top of the tank),

The "coin" of oil (diameter y) would have a volume of:

$\Delta V = \pi\, r^2\, h$ (*h would be the thickness of the coin*)

$\Delta V = \pi \left(\frac{1}{2}y\right)^2 \Delta y$ *where* Δy *is the thickness of the coin*

$\Delta V = \frac{\pi}{4}\, y^2\, \Delta y$



Note: Weight = $\binom{\text{weight per}}{\text{unit vol.}}$ × volume

$$F(y) = 57\,\Delta V = \frac{57\pi}{4} y^2\,\Delta y \text{ lbs}$$

The distance which F(y) must act to lift the coin to the level of the rim is about (10–*y*) *ft, so the work done lifting the coin is about*

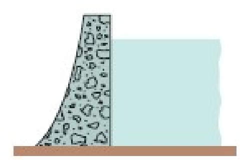
$$\Delta W = \frac{57\pi}{4}(10 - y)y^2\Delta y \text{ ft} \cdot \text{lb}$$

$$W = \sum \frac{57\pi}{4}(10 - y)y^2\Delta y \quad \rightarrow \quad W = \frac{57\pi}{4}\int_0^8 (10 - y)y^2\,dy = \frac{57\pi}{4}\int_0^8 (10y^2 - y^3)\,dy$$

$$W = \frac{57\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 = 30{,}561\, ft \cdot lb$$

Fluid Force and Pressure

Dams are thicker at the bottom that at the top due to water pressure increasing as you go deeper. The pressure at any point on the dam depends only on how far below the surface the point lies and not on how much water the dam is holding back. Fluid pressure *p* (force per unit area) at depth *h* is:

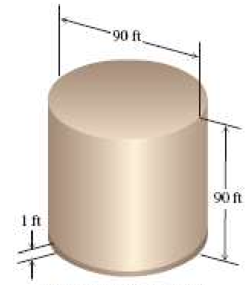


$$p = wh$$

Units: $\frac{lbs}{ft^2}$

w – weight density of the liquid (weight per volume unit)

Ex. At 1:00pm on Jan 15, 1919, a 90 ft high, 90 ft diameter cylindrical metal tank in which Puritan Distilling Company stored molasses at the corner of Foster and Commercial streets in Boston's North End exploded. Molasses flooded the streets 30 feet deep, trapping pedestrians and horses, knocking down buildings, and oozing into homes. It was eventually tracked all over town and even made its way into the suburbs via trolley cars and people's shoes. It took weeks to clean up.



SHADED BAND NOT TO SCALE

(a) Given that the tank was full of molasses weighing 100 lb/ft^3, what was the total force exerted by the molasses on the bottom of the tank at the time it ruptured?
(b) What was the total force against the bottom foot-wide band of the tank wall?

SOLUTION:

(a) At the bottom of the tank, the molasses exerted a constant pressure of:

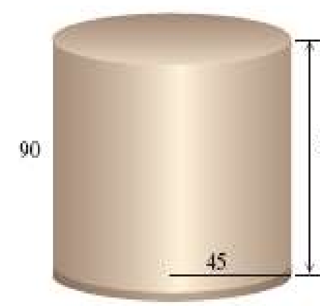
$$p = wh = \left(100\ \frac{lbs}{ft^3}\right)(90\ ft) = 9000\ \frac{lbs}{ft^2}$$

Since the area of the base was $\pi(45)^2 = 2025\pi\ ft^2$, the total force on the base was:

$p = (9000)(2025\pi) = 57{,}225{,}526$ lbs.

(b) We partition the band from depth 89 feet to depth 90 feet into narrower bands of width Δy and choose a depth y_k in each one. The pressure at this depth is $p = wh = 100y_k$ lbs/ft^2. The force against each narrow band is approx.

$pressure \times\ area = (100y_k)(90\pi\Delta y) = 9000\pi y_k\Delta y$ lbs.



The 1-ft band at the bottom of the tank wall can be partitioned into thin strips on which the pressure is approximately constant.

Adding the forces against all the bands leads us to:

$$F = \int_{89}^{90} 9000\pi y\ dy = 9000\pi \left[\frac{y^2}{2}\right]_{89}^{90} \approx 2{,}530{,}533 \text{ lbs.}$$

Weight vs. Mass

Weight is the force that results from gravity pulling on a mass. The two are related by the equation in Newton's second law,

$$\text{weight} = \text{mass} \times \text{acceleration}.$$

Thus,

$$\text{newtons} = \text{kilograms} \times \text{m/sec}^2,$$
$$\text{pounds} = \text{slugs} \times \text{ft/sec}^2.$$

To convert mass to weight, multiply by the acceleration of gravity. To convert weight to mass, divide by the acceleration of gravity.

Homework: pg. 425 #1-5, 7, 8, 10–14, 17, 18

7.5b Applications from Science and Statistics (Day 2)

Normal Probabilities

Question: Suppose you find an old clock in the attic. What is the probability that is has stopped between 2:00 and 5:00?

This is not your typical probability question because the sample space is infinite in size. Time is not a finite value. But if you at a clock face, the region of the clock between 2:00 and 5:00 is a part of a circle (a sector actually) and it has an ***AREA****! Therefore we can use integration to help find this answer.*

Def: **Probability Density Function (PDF)**

A **probability density function** is a function $f(x)$, with domain all reals such that

$f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (remember, sum of all probabilities of an event is 1)

Then, the probability associated with an interval $[a, b]$ is

$$\int_a^b f(x)\,dx$$

Ex. Find the probability that the clock stopped between 2:00 and 5:00.

$$f(t) = \begin{cases} \frac{1}{12}, & 0 \leq t \leq 12 \\ 0, & \text{otherwise} \end{cases}$$

The probability that the clock stopped at some time t with $2 \leq t \leq 5$ *is*

$$\int_2^5 f(t)dt = 3 \times \frac{1}{12} = \frac{1}{4}$$

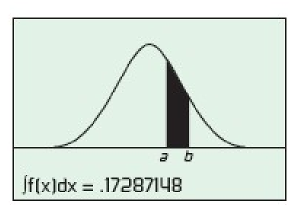
There are three sectors of the clock's 12 hours sectors associated with 2:00–5:00

recall: The Normal Curve, also known as the Bell Curve, enables us to describe entire populations based on statistical measurements taken from a sample of the population

Measurements needed:

(1) The mean (μ or $\bar{x}$)
- Average value of x

(2) The standard deviation (σ or s)
- The scatter measurement around the mean



A normal probability density function
The probability associated with the interval [a,b] is the area under the curve

Def: Normal Probability Density Function (pdf)

The normal probability density function (The Gaussian Curve) for a population with mean μ and standard deviation σ is

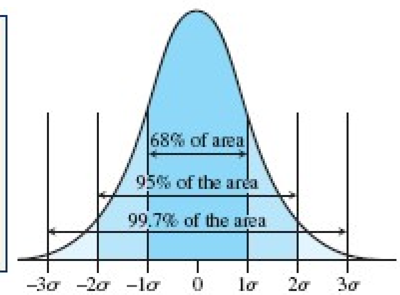
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}$$

Discovered by Karl Friedrick Guass

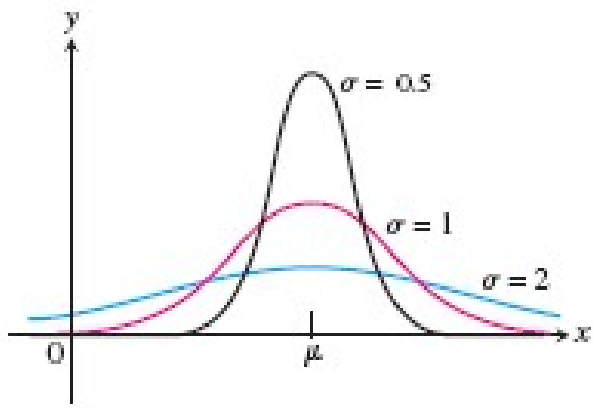
The 68-95-99.7 Rule for Normal Distributions

Given a normal curve,

- 68% of the area will lie within σ of the mean μ,
- 95% of the area will lie within 2σ of the mean μ,
- 99.7% of the area will lie within 3σ of the mean μ.



Even with this rule, the area under the curve can spread out quite a bit, depending on the size of σ. Below shows three normal pdfs with mean μ = 2 and standard deviations σ = 0.5, 1, and 2:



Suppose a telephone help line takes a mean of 2 minutes to answer calls. If the standard deviation is $\sigma = 0.5$, then (according to the blue curve above), 68% of the calls are answered in the range of 1.5 to 2.5 minutes ($\pm 1\sigma$) and 99.7% of the calls are answered in the range of 0.5 to 3.5 minutes ($\pm 3\sigma$).

Ex. Suppose frozen spinach boxes marked as 10 oz of spinach have a mean weight of 10.3 oz and a standard deviation of 0.2 oz.

(a) What percentage of *all* such spinach boxes can be expected to weigh between 10 and 11 oz?
(b) What percentage would we expect to weigh less than 10 oz?
(c) What is the probability that a box weighs *exactly* 10 oz?

(a) Using an integral to get the sum of all: $\int_{10}^{11} \frac{1}{0.2\sqrt{2\pi}}\, e^{-(x-10.3)^2/(2(0.2)^2)}dx \approx 0.933 \;\rightarrow\; 93.3\%$

(b) With the mean at 10.3 and a standard deviation of 0.2, the curve will approach the *x*-axis pretty quickly. Therefore at 9 oz, the area under the curve is close to 0.000000001, so we will use 9 as our lower bound:

$$\int_{9}^{10} \frac{1}{0.2\sqrt{2\pi}}\, e^{-(x-10.3)^2/(2(0.2)^2)}dx \approx 0.067 \;\rightarrow\; 6.7\%$$

(c) $\int_{10}^{10} \frac{1}{0.2\sqrt{2\pi}}\, e^{-(x-10.3)^2/(2(0.2)^2)}dx = 0 \;\rightarrow\; 0\%$ why? Because 10 is only one of an infinite number of weights

Homework: pg. 425–428 #6, 9, 15, 16, 20, 24, 27–31, 35–39