Chapter 6 – Differential Equations and Mathematical Modeling

6.1 Antiderivatives and Slope Fields

> Def: An equation of the form:
> $$\frac{dy}{dx} = y \ln x$$
> which contains a derivative is called a ***Differential Equation***.

- In this equation, you are to find a function y in terms of x (i.e. $y = f(x)$)
- When we are given the derivative of a function and its value at a given point:

  Ex. $\dfrac{dy}{dx} = 2x - 2 \quad f(4) = 10$

  This is called an ***initial value problem***.

  - The value of f for one value of x is the initial condition of the problem.
- Finding ALL functions y that satisfy the differential equation is called **solving the differential equation.**

Ex. Find all functions y that satisfy $\frac{dy}{dx} = \sec^2 x + 2x + 5$

This is a simple antiderivative question: $y = \tan x + x^2 + 5x + C$

The " $+ C$ " at the end presents a constant. Since C could be any value, then there are an infinite number of solutions. This is referred to as a **FAMILY OF SOLUTIONS**.

Ex. Find the particular solution to the equation $\frac{dy}{dx} = e^x - 6x^2$ whose graph passes through (1,0).

Using antiderivatives: $y = e^x - 2x^3 + C$
At (1,0): $0 = e^1 - 2(1) + C \quad \rightarrow \quad C = 2 - e$
The particular solution is: $y = e^x - 2x^3 + 2 - e$

Ex. Find the particular solution to the equation $\frac{dy}{dx} = 2x - \sec^2 x$ whose graph passes through (0,3).

The family of solutions is: $y = x^2 - \tan x + C$.
Applying the condition – at (0,3): $3 = 0 - \tan 0 + C \quad \rightarrow \quad C = 3$
The particular solution is $y = x^2 - \tan x + 3$

Since tan is undefined at $\pm\frac{\pi}{2}$, we add the domain stipulation $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Ex. Find the solution to the differential equation $f'(x) = e^{-x^2}$ for which $f(7) = 3$.

Using the Fundamental Theorem of Calculus: $f(x) = \int_7^x e^{-t^2} dt + 3$
This results in $f(7) = \int_7^7 e^{-x^2} dt + 3 = 0 + 3 = 3$

If we want $f(-2)$, then $f(-2) = \int_7^{-2} e^{-t^2} dt + 3$ Using a calculator we get 1.2317

Ex. Suppose $100 is invested in an account that pays 5.6% interest compounded continuously. Find a formula for the amount in the account at any time *t*.

Because the interest is compounded on the original value, the equation would be:

$$\frac{dy}{dx} = 0.056y \quad \text{with } y(0) = 100$$

How do you solve it?

- Well one way is to think of a function whose derivative is a multiple of itself.
- Another is to get the multiple of itself.
    - This would be the exponential function:

$$y(t) = Ce^{0.056x}$$

Proof:

$$y = Ce^{0.056x}$$

$$\frac{dy}{dx} = C(0.056e^{0.056x}) = 0.056(Ce^{0.056x}) = 0.056y(t)$$

Using the pre-existing condition:

$$y(t) = Ce^{0.056x}$$

$$y(0) = Ce^{0.056(0)}$$

$$100 = Ce^{0}$$

$$100 = C$$

- This leads to the equation: $y = 100e^{0.056x}$

- The initial solution $y(t) = Ce^{0.056t}$ is the family of solutions. Depending on the value of C.
    - Because of the pre-existing condition $y(0) = 100$, we could find the value of *C*.

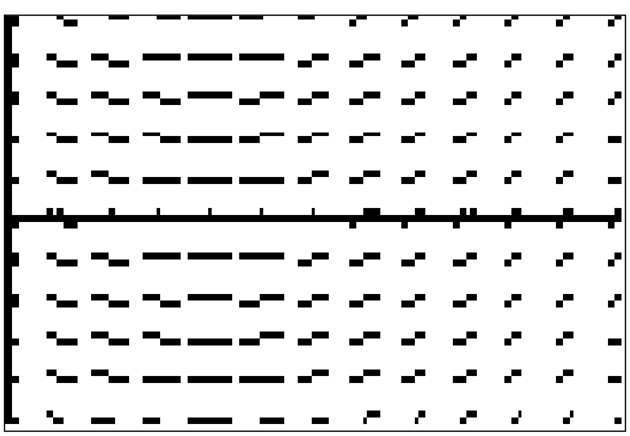
- Is there a way to see the family of solutions for a particular integral?? Yes, it is called Slope Fields.

<u>Def</u>: **Slope Field**

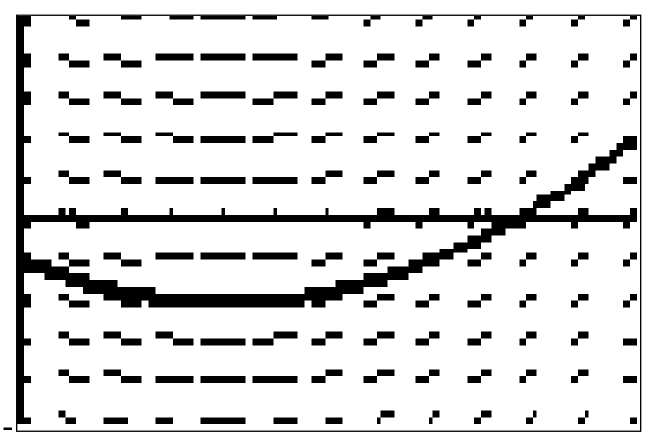
A <u>slope field</u> (or directional field) for a differential equation

$$\frac{dy}{dx} = f(x, y)$$

is a plot of short line segments with slopes $f(x, y)$ for a lattice (set) of points (x, y) in the plane.



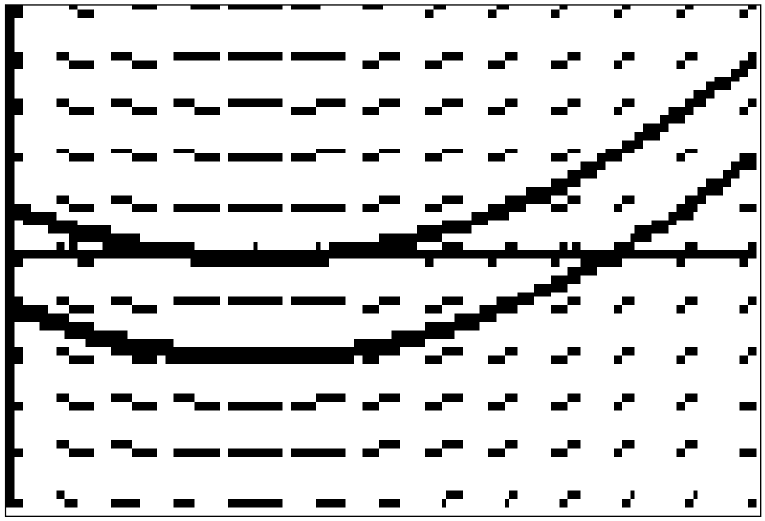
Slope field for $\frac{dy}{dx} = 2x - 2$



Same slope field with $y = x^2 - 2x - 1$

- Each of the little lines in the slope field represent a tangent line to one of the family of curves for the differential equation.

- Here is the same slope field with two curves: $y = x^2 - 2x - 1$ and $y = x^2 - 2x + 1$

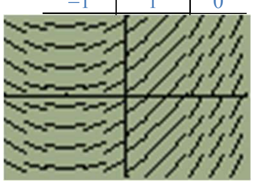


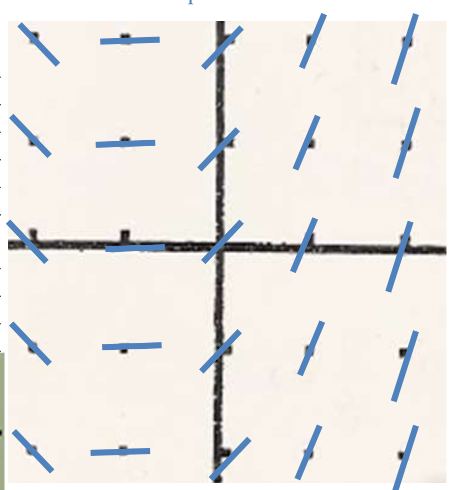
Draw a slope field for each of the following differential equations. Each tick mark is one unit.

1. $\frac{dy}{dx} = x + 1$

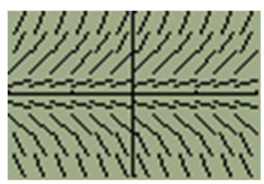
Make a table with all the points (can only show some). Plug the x and y into the derivative and find its value. Draw a mini line with that slope.

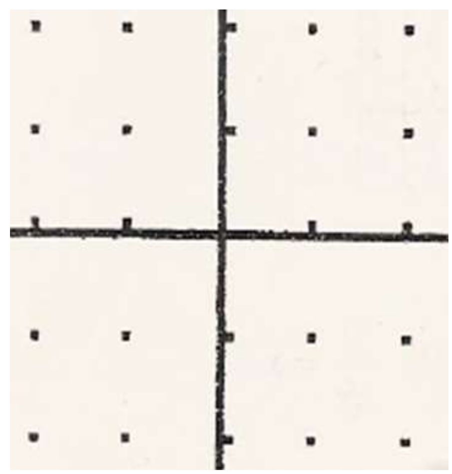
x	y	$\frac{dy}{dx}$
−2	−2	−1
−2	−1	−1
−2	0	−1
−2	1	−1
−2	2	−1
−1	−2	0
−1	−1	0
−1	0	0
−1	1	0



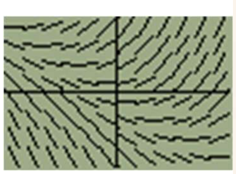


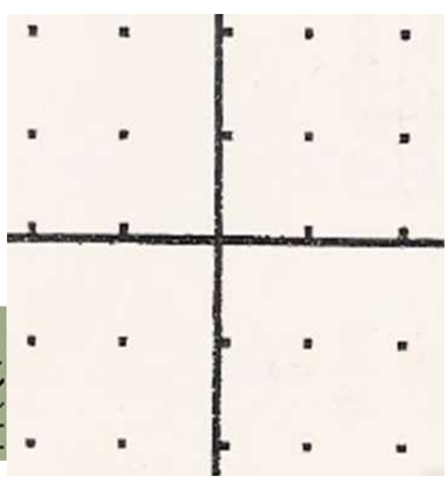
2. $\frac{dy}{dx} = 2y$



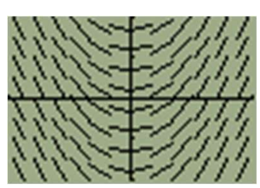


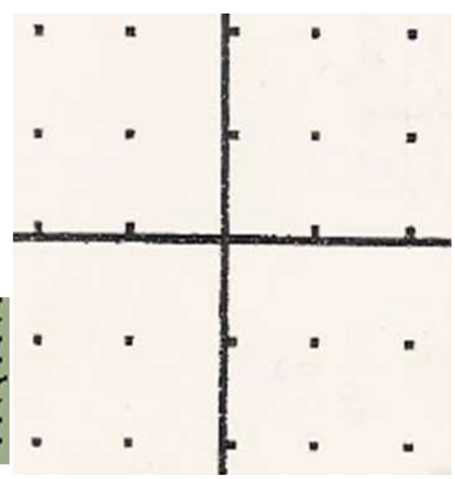
3. $\frac{dy}{dx} = x + y$



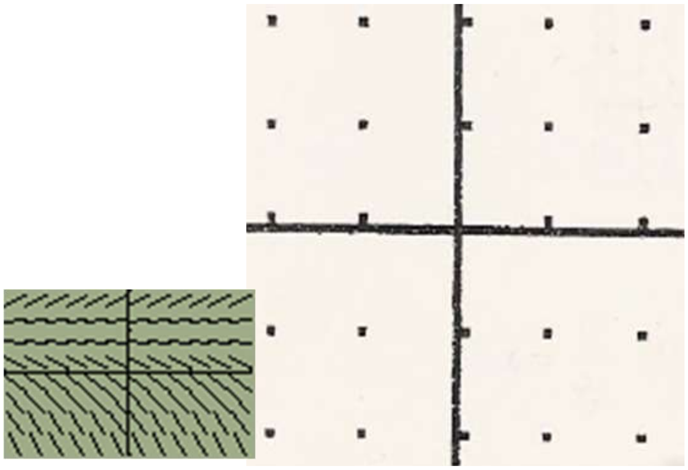


4. $\frac{dy}{dx} = 2x$

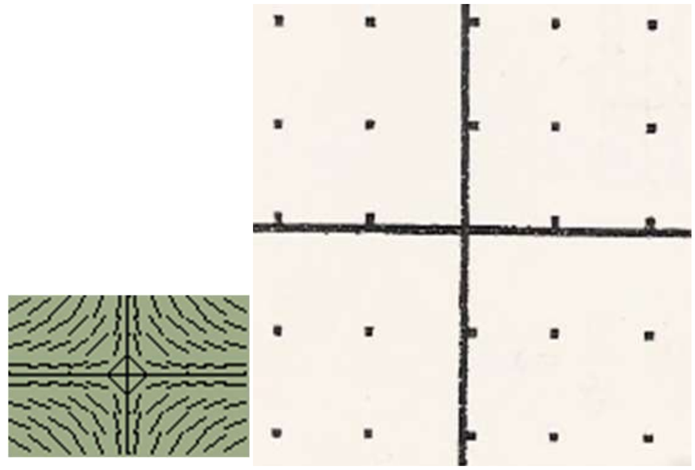




5. $\frac{dy}{dx} = y - 1$

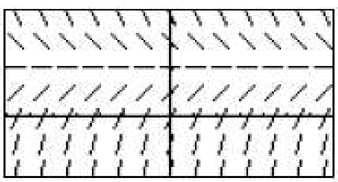


6. $\frac{dy}{dx} = -\frac{y}{x}$

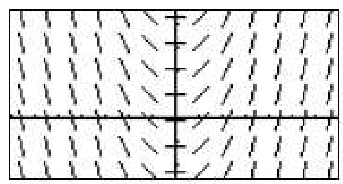


Match the slope fields with their differential equations.

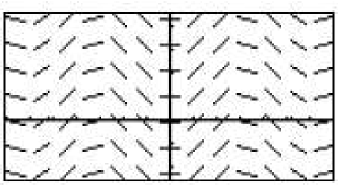
(A)



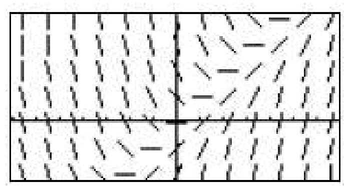
(B)



(C)



(D)



7. $\frac{dy}{dx} = \sin x$ 8. $\frac{dy}{dx} = x - y$ 9. $\frac{dy}{dx} = 2 - y$ 10. $\frac{dy}{dx} = x$

C **D** **A** **B**

<u>**Homework**</u>**: Pg. 327-329 #1-3, 5, 7-10, 12-24(3's), 25-39odd, 51, 56**

6.2 Antidifferentiation by Substitution

Def: ***Indefinite Integral***

The set of all antiderivatives of a function $f(x)$ is the ***indefinite integral*** of f with respect to x and is denoted by:

$$\int f(x)dx$$

And if $F(x)$ is an antiderivative of $f(x)$ as defined in the Fundamental Theorem, then

$$\int f(x)dx = F(x) + C$$

where $F'(x) = f(x)$ and C is ANY constant value.

- C is called the ***constant of integration*** and is an arbitrary constant

Ex. Evaluate:

1. $\int 2x dx = x^2 + C$

2. $\int \cos x dx = \sin x + C$

3. $\int (x^2 + 4x - 5)dx$

   $\frac{x^3}{3} + 2x^2 - 5x + C$

4. $\int e^x dx$

   $e^x + C$

5. $\int \frac{1}{\sqrt{x}}dx \quad = \int x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}} + C$

- Below is a table of some basic Indefinite Integrals:

$\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$	$\int \frac{dx}{x} = \ln\|x\| + C$	$\int e^{kx} dx = \frac{e^{kx}}{k} + C$
$\int \sin kx dx = -\frac{\cos kx}{k} + C$	$\int \cos kx\, dx = \frac{\sin kx}{k} + C$	$\int \sec^2 kx dx = \frac{\tan kx}{k} + C$
$\int \csc^2 kx dx = -\frac{\cot kx}{k} + C$		

ex. Evaluate: $\int e^{-4x} dx$ $\qquad$ $\int \cos \frac{1}{2}x dx$

$-\frac{1}{4}e^{-4x} + C$ $\qquad$ $2\sin \frac{1}{2}x + C$

Properties of Indefinite Integrals:

Let k be a real number:

1. *Constant Multiple Rule:* $\int kf(x)dx = k\int f(x)dx$

   If $k=-1$, *then,* $\int -f(x)dx = -\int f(x)dx$

2. *Sum and Difference Rule:* $\int (f(x)\pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

Ex. Evaluate $\int (3x^2-4x+5)dx$

We can use the two rules above to evaluate this:

$$\begin{aligned}\int (3x^2-4x+5)dx &= \int 3x^2dx - \int 4xdx + \int 5dx\\ &= 3\int x^2dx - 4\int xdx + 5\int dx\\ &= 3\cdot\left(\frac{x^3}{3}+C_1\right) - 4\left(\frac{x^2}{2}+C_2\right) + 5(x+C_3)\\ &= x^3+3C_1-2x^2-4C_2+5x+5C_3\\ &= x^3-2x^2+5x+(3C_1-4C_2+5C_3)\\ &= x^3-2x^2+5x+C\end{aligned}$$

Ex. A heavy projectile is fired straight up into the air from a platform 3 meters above the ground with an initial velocity of 160 m/sec. Assume that only the force affecting the projectile during its flight is gravity, which produces a downward acceleration of 9.8 m/sec^2. If $t=0$ when the projectile is fired, find a formula for the projectile's

(a) velocity as a function of time t

(b) height above the ground as a function of time t.

$v(0)=160$
$s(0)=3$
$a(t)=-9.8$

The velocity function is the derivative of the position function

(a) $v(t)=\int a(t)\,dt = \int -9.8\,dt = -9.8t + C$

$v(0) = -9.8(0)+C = 160 \quad \rightarrow \quad C=160$

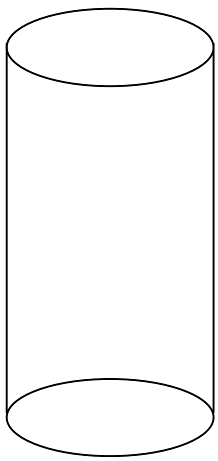
$v(t) = -9.8t + 160$

(b) $s(t) = \int v(t)\,dt = \int -9.8t+160\,dt = -4.9t^2+160t+C$

$s(0) = -4.9(0)+160(0)+C = 3 \quad \rightarrow \quad C=3$

$s(t) = -4.9t^2+160t+3$

ex. A right cylindrical tank with radius 5 ft and height 16 ft was initially filled with water is being drained at a rate of $0.5\sqrt{x}$ ft^3/min, where x is the depth of the water. Find a formula for the depth and the amount of water in the tank at any time t. How long will it take the tank to empty?



Volume of a Cylinder: $V = \pi r^2 h$

Initial value: $h(0) = 16$

at any given height h, the volume of this tank would be:

$$V = \pi 5^2 h = 25\pi h$$

Taking the derivative with respect to time:

$$V = 25\pi h$$

$$\frac{dV}{dt} = -25\pi \frac{dh}{dt}$$

Negative due to draining

$$0.5\sqrt{h} = -25\pi \frac{dh}{dt}$$

Solving for *dh/dt*:

$$\frac{0.5\sqrt{h}}{-25\pi} = \frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{50\pi}$$

We can now solve this differential equation:

Remember:

1. Get the h's with the dh (× ÷)
2. Move the *dt* to the other side.

Both sides should be integrals you can do!!

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{50\pi}$$

$$\frac{1}{\sqrt{h}}\frac{dh}{dt} = -\frac{1}{50\pi}$$

$$h^{-\frac{1}{2}}\frac{dh}{dt} = -\frac{1}{50\pi}$$

$$\int h^{-\frac{1}{2}}\frac{dh}{dt}\,dt = \int -\frac{1}{50\pi}\,dt$$

$$\int h^{-\frac{1}{2}}\,dh = \int -\frac{1}{50\pi}\,dt$$

$$2h^{\frac{1}{2}} + C_1 = -\frac{1}{50\pi}t + C_2$$

$$2\sqrt{h} = -\frac{1}{50\pi}t + C$$

Using the initial condition: $h(0) = 16$

$h = 16 \qquad t = 0$

$$2\sqrt{16} = -\frac{1}{50\pi}(0) + C$$

$$8 = C$$

:

Substituting back into the equation, we get

$$2\sqrt{h} = -\frac{1}{50\pi}t + 8$$

$$\sqrt{h} = -\frac{1}{100\pi}t + 4$$

$$h = \left(4 - \frac{1}{100\pi}t\right)^2$$

Substituting back into the Volume formula:

$V = 25\pi h$

$$V = 25\pi\left(4 - \frac{1}{100\pi}t\right)^2$$

Homework: Pg 337-338 QR #1-10; Ex #1-10, 17-24

6.2b Integration by Substitution (Day 2)

Ex. Evaluate

1. $\int x^5 dx$ 2. $\int 4x^3 - 3x dx$ 3. $\int e^x - \cos x \, dx$

$\frac{x^6}{6} + C$ $x^4 - \frac{3x^2}{2} + C$ $e^x - \sin x + C$

4. $\int (x+3)^2 dx$

$\int x^2 + 6x + 9 \, dx = \frac{x^3}{3} + 3x^2 + 9x + C$

$(x+2)^5 =$
$x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$
$\int x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32 \, dx$
$\frac{x^6}{6} + 2x^5 + 10x^4 + \frac{80}{3}x^3 + 40x^2 + 32x + C$

$\frac{x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + C}{6}$

UGH! Lots of work!

Would be a lot easier if it had been x^5

What if you had $\int (x+2)^5 dx$? Do you multiply it out?

Suppose, we use a substitution of:

Let $u = x + 2$

$\frac{du}{dx} = 1$

$du = dx$

We have dx and will need du when we change the variable.

So, take the derivative of u with respect to x

So, we now have:

$\int u^5 du = \frac{u^6}{6} + C$ but $u = x + 2$, so we get $\frac{(x+2)^6}{6} + C$ But is this the same as

$(x+2)^6 = x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$ YEP!

Note the 64 is replaced by +C (a constant!!)

- When u is a differentiable function of x and n is a real number not equal to -1, then the Chain Rule gives us:

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}$$

- Switch the direction, we get:

This is a proof on why this type of substitution works!

$$\int \left(u^n \frac{du}{dx}\right) dx = \frac{u^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

- A change in variable can often make an unfamiliar or difficult looking integral into one that we can evaluate.
  - This method is called ***substitution method of integration***.

To evaluate integrals that are not "simple":

1. Let u = an expression whose derivative in also in the expression (or a multiple of)
2. Take the derivative of the equation written in step 1, and solve for du
3. It should be the other expression in the integrand and dx
4. Make you substitutions and integrate with respect to u (this should be a "simple" integral).
5. Substitute back into the u and add a "+C" if it is an indefinite integral.

Ex. Evaluate: $\int \cos(3x-2)dx$

“Aside”

Let $u = 3x - 2$

$\frac{du}{dx} = 3$

$du = 3dx$

$\frac{1}{3}du = dx$

$\int \cos u \left(\frac{1}{3}\, du\right)$

$\frac{1}{3}\int \cos u \;du$

$\frac{1}{3}\sin u$

Substitute back in

$\frac{1}{3}\sin(3x-2) + C$

STEPS

1. Let u = an expression whose derivative in also in the expression (or a multiple of)
2. Take the derivative of the equation written in step 1, and solve for du
3. It should be the other expression in the integrand and dx
4. Make you substitutions and integrate with respect to u (this should be a “simple” integral).
5. Substitute back into the u and add a “$+C$” if it is an indefinite integral.

First rewrite using trig formulas

Ex. Evaluate $\int \frac{1}{\cos^2 2x}dx = \int \sec^2 2x\, dx$

“Aside”

Let $u = 2x$

$\frac{du}{dx} = 2$

$du = 2dx$

$\frac{1}{2}du = dx$

You can skip the indicated step by simply using the following:

$du = \frac{du}{dx} \cdot dx$

If there is a constant in front of du, just put it in front of the integral

$\frac{1}{2}\int \sec^2 u\, du$

$\frac{1}{2}\tan u$

Substitute back in

$\frac{1}{2}\tan 2x + C$

First rewrite using trig formulas

Ex. Evaluate $\int \tan x dx = \int \frac{\sin}{\cos x}dx$

“Aside”

Let $u = \cos x$

$du = -\sin x\, dx$

$-du = \sin x\, dx$

$-\int \frac{1}{u}du$

$-\ln|u|$

$-\ln|\cos x| + C$

This one is different because it’s not a constant times dx when you differentiate.

You had better have the extra expression with the dx in the integrand.

Most of the time, you will want to let u be the denominator!!

Ex. Evaluate: $\int \sin^3 x \cos x dx$

“Aside”

Let $u = \sin x$

$du = \cos x\, dx$

$du = \cos x\, dx$

$\int u^3 du$

$\frac{u^4}{4}$

$\frac{\sin^4 x}{4} + C$

In this case, we didn’t have a denominator to let u equal.

So, in this case, use the expression that is being raised to the power.

There is no real set way to choose. BUT YOU WILL KNOW QUICKLY IF YOU MESSED UP

$\int (x^2+2x-3)^2(x+1)dx$

“Aside”

Let $u = x^2 + 2x - 3$

$du = (2x+2)dx$

$du = 2(x+1)dx$

$\frac{1}{2}\,du = (x+1)dx$

$\frac{1}{2}\int u^2\, du$

$\frac{1}{2}\cdot\frac{u^3}{3}$

$\frac{u^3}{6}$

$\frac{(x^2+2x-3)^3}{6} + C$

- What if it a definite integral? Same process – just plug the value of u back in then use the bounds.

Ex. Evaluate $\displaystyle\int_0^{\pi/4} \tan x \sec^2 x\, dx$

"Aside"

Let $u = \tan x$
$du = \sec^2 x\, dx$

$\int u\, du$

$\frac{u^2}{2}$

$$\left.\frac{(\tan x)^2}{2}\right|_0^{\frac{\pi}{4}} = \frac{\left(\tan\frac{\pi}{4}\right)^2 - (\tan 0)^2}{2} = \frac{1-0}{2} = \frac{1}{2}$$

Ex. Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$

"Aside"

Let $u = x^3 + 1$

$du = 3x^2 dx$

$$\int \sqrt{u}\, du = \int u^{\frac{1}{2}}\, du$$

$$\frac{2}{3}u^{\frac{3}{2}} = \left.\frac{2}{3}\sqrt{(x^3+1)^3}\right|_{-1}^{1}$$

$$\frac{2}{3}\left[\sqrt{(1+1)^3} - \sqrt{(-1+1)^3}\right] = \frac{2}{3}\left[\sqrt{8} - 0\right] = \frac{2}{3}\sqrt{8}$$

- There are some people who think that when you make the u-substitution, that you change the bounds according the substitution.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Ex. Write as an equivalent integral

1. $\int_0^5 x^2(x^3-1)\, dx$

Let $u = x^3 - 1$ $5 \to 124$
$du = 3x^2 dx$ $0 \to -1$
$\frac{1}{3}du = x^2 dx$

$$\frac{1}{3}\int_{-1}^{124} u\, du = \frac{1}{3}\cdot\left.\frac{u^2}{2}\right|_{-1}^{124} = \frac{(124)^2 - 1}{6} = \frac{15375}{6}$$

2. $\int_4^8 \frac{\ln x}{x} dx$

"Aside"

Let $u = \ln x$ $8 \to \ln 8$
$du = \frac{1}{x} dx$ $4 \to \ln 4$

$$\int_{\ln 4}^{\ln 8} u\, du = \left.\frac{u^2}{2}\right|_{\ln 4}^{\ln 8} = \frac{1}{2}\left[(\ln 8)^2 - (\ln 4)^2\right]$$

$$\frac{1}{2}(\ln 8)^2 - \frac{1}{2}(\ln 4)^2$$

3. $\int_{\pi/6}^{\pi/3} \tan^7 x \sec^2 x\, dx$

"Aside"

Let $u = \tan x$ $\pi/3 \to \sqrt{3}$
$du = \sec^2 x\, dx$ $\pi/6 \to \sqrt{3}/3$

$$\int_{\sqrt{3}/3}^{\sqrt{3}} u^7 du = \left.\frac{u^8}{8}\right|_{\sqrt{3}/3}^{\sqrt{3}} = \frac{(\sqrt{3})^8 - \left(\frac{1}{3}\sqrt{3}\right)^8}{8} = \frac{81 - \frac{81}{6561}}{8} = \frac{531360}{52488}$$

Homework: Pg. 337 – 338 QR 1-10, Ex. 1 – 8, 9 – 29odd, 32, 33, 39, 42, 43

6.2c Separable Differential Equations

- A differential equation $y' = f(x, y)$ is ***separable*** if *f* can be expressed as a product of a function of *x* and a function of *y*.
  - This means:

$$\frac{dy}{dx} = g(x)h(y)$$

- If $h(y) \neq 0$, then we can **separate the variables** by dividing both sides of the equation by $h(y)$:

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

- Integrating both sides of the equation with respect to *x*, we get:

$$\int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)dx$$

$$\int \frac{1}{h(y)}dy = \int g(x)dx$$

- With the variables separated, we can integrate each side of the equation and write the function expressing *y* as a function of *x*.

Ex. Solve the differential equation

$$\frac{dy}{dx} = 2x(1+y^2)e^{x^2}$$

Separate the variables

$$\frac{1}{(1+y^2)}dy = 2xe^{x^2}dx$$

$$\int \frac{1}{(1+y^2)}dy = \int 2xe^{x^2}dx$$

"Aside"

Let $u = x^2$

$du = 2xdx$

$\int e^u du = e^u$

e^{x^2}

$$\tan^{-1} y = e^{x^2} + C$$

$$y = \tan\left(e^{x^2} + C\right)$$

Ex. Solve the differential equation

$$\frac{dy}{dx} = x\sqrt{y}\cos^2\sqrt{y}$$

Separate the variables

$$\frac{1}{\sqrt{y}\cos^2\sqrt{y}}dy = x\,dx$$

$$\frac{\sec^2\sqrt{y}}{\sqrt{y}}dy = x\,dx$$

$$\int \frac{\sec^2\sqrt{y}}{\sqrt{y}}dy = \int x\,dx$$

"Aside"

Let $u = \sqrt{y}$

$du = \frac{1}{2}y^{-\frac{1}{2}}dy$

$2du = \frac{1}{\sqrt{y}}dy$

$2\int \sec^2 u\,du$

$2\tan u = 2\tan\sqrt{y}$

$$2\tan\sqrt{y} = \frac{1}{2}x^2 + C$$

$$\tan\sqrt{y} = \frac{1}{4}x^2 + C$$

$$\sqrt{y} = \tan^{-1}\left(\frac{x^2+C}{4}\right)$$

$$y = \left[\tan^{-1}\left(\frac{x^2+C}{4}\right)\right]^2$$

Homework: Day 1: Pg 337 – 338 #25– 29, 32, 33, 39, 41, 43, 47, 45-66(3s)
Day 2 Worksheet: Integration by Substitution

6.3 Integration by Parts

- There are some integrals that can't be done by a simple *u*-substitution.
  - So we need another way to find these integrals.
- Suppose we take the Product Rule and write it in integral form:

$$\frac{d}{dx}(uv) = uv' + vu'$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

- Integrating both sides with respect to *x* and rearranging:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$uv = \int udv + \int vdu$$

$$\boxed{\int udv = uv - \int vdu}$$

- This formula is called ***Integration by Parts***
  - This formula expresses one integral $\int udv$ in terms of another $\int vdu$
    - Choosing the correct *u* and *v* may make the second integral ($\int vdu$) easier to evaluate.

Ex. Evaluate $\int x\cos x dx$

$u = x \qquad dv = \cos x\, dx$

$du = dx \qquad v = \sin x$

$uv - \int v\, du$

$x\sin x - \int \sin x\, dx$

$x\sin x + \cos x + C$

To Set up Integration by Parts:
1. You want to split this integral up into a *u* and a *dv*.
   a. *u* should be easy to find the derivative
   b. *dv* should be an easy integral
2. Find the derivative of *u* and integrate *dv*
3. Plug into the formula
   $\int udv = uv - \int vdu$
4. Simplify/evaluate

- You are trying to make the second integral easy to do. One method selecting a *u* is using the following "Order of Selection":
  1. Natural Log (ln) **L**
  2. Inverse Trig Function **I**
  3. Polynomial Function **P**
  4. Exponential Function **E**
  5. Trigonometry Function **T**

L I P E T

- This is strictly a guide (works 95% of time).

Ex. Find the area bounded by the curve $y = xe^x$ and the *x*-axis from $x = 0$ to $x = 3$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$uv - \int vdu$

$xe^x - \int e^x\, dx$

$xe^x - e^x \Big|_0^3 = (3e^3 - e^3) - (0 - 1) = \quad 2e^3 + 1$

Ex. Evaluate: $\int \ln x dx$

$u = \ln x \qquad dv = dx$
$du = \frac{1}{x}dx \qquad v = x$

$uv - \int v\,du$

$x\ln x - \int x\frac{1}{x}dx$

$x\ln x - \int 1dx$

$x\ln x - x + C$

Ex. Evaluate: $\int_0^1 \tan^{-1} x dx$

$u = \tan^{-1} x \qquad dv = dx$
$du = \frac{1}{1+x^2}dx \qquad v = x$

$uv - \int v\,du$

$x\tan^{-1} x - \int \frac{x}{1+x^2}dx$

Let $u = 1 + x^2 \quad \rightarrow \quad du = 2xdx \quad \rightarrow \quad \frac{1}{2}du = dx$

$x\tan^{-1} x - \frac{1}{2}\int \frac{1}{u}du$

$x\ \tan^{-1} x - \frac{1}{2}\ln|u|$

$x\tan^{-1} x - \frac{1}{2}\ln|1 + x^2| + C$

$x\tan^{-1} x - \frac{1}{2}\ln(1 + x^2) + C$

- Sometimes, you need to play around a little with the integral:

Ex. Evaluate: $\int e^x \cos x dx$

$u = e^x \qquad dv = \cos x\,dx$
$du = e^x dx \qquad v = \sin x$

By Parts

$uv - \int v\,du$

$\int e^x \cos\ x\,dx = e^x \sin x - \int e^x \sin x\,dx$

$u = e^x \qquad dv = \sin x\,dx$
$du = e^x dx \qquad v = -\cos x$

$\int e^x \cos\ x\,dx = e^x \sin x - \left[-e^x \cos x - \int -e^x \cos x\,dx\right]$

$\int e^x \cos\ x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos\ x\,dx$

$2\int e^x \cos\ x\,dx = e^x \sin x + e^x \cos x$

$\int e^x \cos\ x\,dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + C$

Homework: Pg. 346 QR #1-10, Ex. 1–8

6.3 Integration by Parts (Part 2)

Ex. Evaluate: $\int x^2 e^x dx$

$u = x^2 \qquad dv = e^x dx$

$du = 2x dx \qquad v = e^x$

By Parts again

$uv - \int v du$

$x^2 e^x - 2\int x e^x \, dx$

$x^2 e^x - 2[x e^x - e^x] = \boxed{x^2 e^x - 2x e^x + 2e^x + C}$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$uv - \int v du$

$x e^x - \int e^x \, dx$

$x e^x - e^x$

Ex. Evaluate: $\int x^3 \sin x dx$

$u = x^3 \quad dv = \sin x \, dx$

$du = 3x^2 \quad v = -\cos x$

$uv - \int v du$

A

$-x^3 \cos x + 3\int x^2 \cos x \, dx$

Parts Again! for $\int x^2 \cos x \, dx$

$u = x^2 \qquad dv = \cos x \, dx$

$du = 2x dx \quad v = \sin x$

$uv - \int v du = x^2 \sin x - \int 2x \sin x \, dx$

Parts again!

$\int 2x \sin x \, dx$

$u = 2x \qquad dv = \sin x \, dx$

$du = 2dx \quad v = -\cos x$

$-2x\cos x + \int 2\cos x \, dx$

$-2x \cos x + 2\sin x$

$x^2 \sin x - [-2x\cos x + 2\sin x]$

$x^2 \sin x + 2x\cos x - 2\sin x$

Tying it all together:

$-x^3 \cos x + 3[x^2 \sin x + 2x\cos x - 2\sin x]$

$\boxed{-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\sin x + C}$

Had to this By Parts 3 times!

Has be a better way!

Tabular Integration

- Many Integration by parts problems are of the form

$$\int f(x)g(x)dx$$

  - in which f can be differentiated repeatedly to become ZERO and g can be integrated repeatedly without difficulty.
- Integration by parts can result in many repetitions to get a solution
- You can organize the calculation (derivatives/integrals) and save work.

Ex. Evaluate: $\int x^2 e^x dx$

Use $f(x) = x^2$ and $g(x) = e^x$

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	(+)	e^x
$2x$	(–)	e^x
2	(+)	e^x
0		e^x

Using this table and combining the products of the functions connected by the arrows and using the alternating signs we get:

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + c$$

Ex. Evaluate $\int x^3 \sin x dx$ using Tabular Integration

x^3	$+$	$\sin x$
$3x^2$	$-$	$-\cos x$
$6x$	$+$	$-\sin x$
6	$-$	$\cos x$
0		$\sin x$

$$-x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x$$

Ex. Evaluate: $\int x^4 e^{2x}\, dx$

x^4	$+$	e^{2x}
$4x^3$	$-$	$\frac{1}{2}e^{2x}$
$12x^2$	$+$	$\frac{1}{4}e^{2x}$
$24x$	$-$	$\frac{1}{8}e^{2x}$
24	$+$	$\frac{1}{16}e^{2x}$
0		$\frac{1}{32}e^{2x}$

$$\frac{1}{2}x^4e^{2x} - x^3e^{2x} + \frac{12}{8}x^2e^{2x} - \frac{24}{16}xe^{2x} + \frac{24}{32}e^{2x} + C$$

$$\frac{1}{2}x^4e^{2x} - x^3e^{2x} + \frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + C$$

Homework **Day 1: Pg. 346 – 347 #9-11, 15-20, 22, 23**
Day 2: Pg. 347 #25, 26a-d, 27, 28, 29, 31, 35

6.4 Exponential Growth and Decay

- Suppose we are interested in the quantity *y,* that increases or decreases at a rate proportional to the amount currently present
  - Population
  - Radioactive element
  - Money
- If we know the amount initially (time t=0), which we will call y_0 (pronounced "Y – naught"), we can find y as a function of t using the initial value of y:
  - <u>Differential Equation to Solve</u>: $\frac{dy}{dx} = ky$
  - <u>Initial Condition</u>: $y = y_0$ when $t = 0$
    - If y is positive and an increasing function, then k is positive, and the rate of growth is proportional to the current value of y.
    - If y is positive and decreasing, then k is negative, and the rate of "decay" is proportion to the amount still left.
  - So, what is the equation for y?

$$\frac{dy}{dt} = ky$$
$$\frac{dy}{y} = kdt$$
$$\ln|y| = kt + C$$
$$e^{\ln|y|} = e^{kt+C}$$
$$|y| = e^C \cdot e^{kt}$$
$$y = \pm Ae^{kt}$$

Using the initial value:

$$y = \pm Ae^{kt}$$
$$y_0 = \pm Ae^{k\cdot 0}$$
$$y_0 = \pm A$$
$$\pm y_0 = A$$

$$y = y_0 e^{kt}$$

<u>The Law of Exponential Change</u>

If y changes at a rate proportional to the amount present ($\frac{dy}{dx} = ky$) and $y = y_0$ when $t = 0$, then

$$y = y_0 e^{kt}$$

If $k > 0$, then this is exponential growth
If $k < 0$, then this is exponential decay

k represents the RATE CONSTANT of the equation

Continuous Compound Interest

- The formula for compound interest on an account with A_0 at a rate of r% over a time period of t years in which there are k times it is compounded is:

$$\boxed{A(t) = A_0\left(1 + \frac{r}{k}\right)^{kt}}$$

- This formula is used when you have a set number of times when the interest is compounded, k.
  - The interest rate never changes so the interest is proportional on the amount that is in the account.

If we were to compound it continuously, then it would be considered an exponential growth. Therefore, the formula would be:

$$\boxed{A(t) = A_0 e^{kt}}$$

You may know it as

$$\boxed{A = Pe^{rt}}$$

Ex. Suppose you have \$5000 is an IRA that earns 6.5% annually. How much would you have after 8 years if the interest was compounded

(a) quarterly (b) monthly (c) daily (d) continuously

(a) $A = P\left(1+\frac{r}{n}\right)^{nt} = 5000\left(1+\frac{0.065}{4}\right)^{4(8)} = \$8{,}375.06$

(b) $A = P\left(1+\frac{r}{n}\right)^{nt} = 5000\left(1+\frac{0.065}{12}\right)^{12(8)} = \$8{,}398.34$

(c) $A = P\left(1+\frac{r}{n}\right)^{nt} = 5000\left(1+\frac{0.065}{365}\right)^{365(8)} = \$8409.75$

(d) $A = Pe^{rt} = 5000e^{0.65(8)} = \$8{,}410.14$

Radioactivity

- Radioactivity works the exact same way as interest, except it is a decay.

  If you have an initial amount y_0 of a radioactive substance, then the amount still present at any later time t will be

$$y = y_0 e^{-kt}$$

- The **half-life** of a radioactive substance is the amount of time for half of the radioactive nuclei present in a sample it decays.

Ex. Find the half-life formula of any radioactive substance as a function of k

$$y = y_0 e^{-kt}$$
$$\frac{1}{2}y_0 = y_0 e^{-k}$$
$$\frac{1}{2} = e^{-kt}$$

$$\ln\frac{1}{2} = \ln e^{-kt}$$
$$-\ln 2 = -kt$$
$$\frac{-\ln 2}{k} = t$$

Half-life gets you the RATE!!!

$$\boxed{k = \frac{\ln 2}{HL}}$$

Ex. If the half-life of Cirpilium is 42 years, how long would it take 15g of Cirpilium to decay to 5g?

HL=42

$$k = \frac{\ln 2}{42}$$
$$k = 0.0165035$$

$$y = y_0 e^{-kt}$$
$$5 = 15e^{-0.0165035t}$$
$$\frac{5}{15} = e^{-0.0165035t}$$
$$\ln\frac{1}{3} = -0.0165035t$$

$$t = 66.5684 \text{ years}$$

Homework: Pg. 357 #1-10, 13,14

6.4 Exponential Growth and Decay (con't)

Ex. Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear reactor accident in 1986. How long will it take for the 10 grams to decay to 1 gram if the half-life of Pu-239 is 24,360 yrs.?

HL=24360

$k = \dfrac{\ln 2}{24360}$

$k = 0.0000284543$

$y = y_0 e^{-kt}$

$1 = 10e^{-0.00002}$

$\dfrac{1}{10} = e^{-0.0000284543t}$

$\ln \dfrac{1}{10} = -\,0.0000284543t$

$-\ln 10 = -\,0.0000284543t$

$t = 80{,}922.17$ years

Ex. Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after day 4. Approximately how many flies were in the original population?

First, let's find the rate using info:
$y = y_0 e^{kt}$

$300 = 100e^{k(4-2)}$
We use the difference in the days to get our *t* value

$3 = e^{2k}$
$\ln 3 = 2k$
$k = 0.5493061443$

Now, using the Day 2 numbers (though you can use Day 4):

$100 = y_0 e^{0.5493061443(2)}$
$100 = y_0(3)$

$y_0 = \dfrac{100}{3} = 33.33333333$ ....

There were approximately 33 flies originally

Ex. Scientists who do carbon-14 dating use 5700 yrs. for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.

Half-life is $5700 = \dfrac{\ln 2}{k}$

HL=5700

$k = \dfrac{\ln 2}{5700}$

$k = 0.000121605$

$y = y_0 e^{-kt}$

If 10% has decayed, then there is 90% present. We assume 100g to start, mean 90g left

$90 = 100e^{-0.000121605t}$
$0.9 = e^{-0.000121605t}$
$\ln 0.9 = -0.000121605t$

$t = 866.418$ years

A common mistake is to use the 10 and not 90. This would have resulted in 18,935 years.

Remember: In the formula y_0 is original amount and y is the amount left!

Newton's Law of Cooling

This is just a variation of the exponential decay function

<u>Theorem</u>: Newton's Law of Cooling (Heating)

If T is the temperature of an object at time t, and T_s, is the surrounding temperature, then

$$T - T_s = (T_0 - T_s)e^{-kt}$$

where T_0 is the original temperature at $t = 0$. k is the rate of cooling.

Ex. Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. If the object cools from 100° to 90° in 10 minutes, how much longer will it take for its temperature to decrease to 80°

1st Sentence:

$\boldsymbol{T = 90}$
$\boldsymbol{T_s = 60}$
$\boldsymbol{T_0 = 100}$
$\boldsymbol{k} = ?$
$\boldsymbol{t = 10}$

You use the first sentence to usually get the value of k:

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$90 - 60 = (100 - 60)e^{-k(10)}$$
$$\frac{30}{40} = e^{-10k}$$
$$\ln 0.75 = -10k$$
$$k = 0.0287682$$

2nd Sentence:

$\boldsymbol{T = 80}$
$\boldsymbol{T_s = 60}$
$\boldsymbol{T_0 = 90}$
$\boldsymbol{k} = 0.02877$
$\boldsymbol{t} = ?$

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$80 - 60 = (90 - 60)e^{-0.02877t}$$
$$20 = 30e^{-0.02877t}$$
$$\frac{2}{3} = e^{-0.02877t}$$
$$\ln\frac{2}{3} = -0.02877t$$
$$t = 14.094$$

About 14 minutes

ex. A deep dish apple pie whose internal temperature is 220°F when removed from the oven, was set on a 40°F porch to cool. Fifteen minutes later, the pie's internal temperature was 180°F. How long did it take the pie to cool from there to 70°F?

1st Sentence:

$\boldsymbol{T = 180}$
$\boldsymbol{T_s = 40}$
$\boldsymbol{T_0 = 220}$
$\boldsymbol{k} = ?$
$\boldsymbol{t = 15}$

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$180 - 40 = (220 - 40)e^{-k(15)}$$
$$\frac{140}{180} = e^{-15k}$$
$$\ln 0.\overline{777} = -15k$$
$$k = 0.0167543$$

2nd Sentence:

$\boldsymbol{T = 70}$
$\boldsymbol{T_s = 40}$
$\boldsymbol{T_0 = 180}$
$\boldsymbol{k} = 0.016754$
$\boldsymbol{t} = ?$

$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$70 - 40 = (180 - 40)e^{-0.016754t}$$
$$30 = 140e^{-0.016754t}$$
$$\frac{3}{14} = e^{-0.016754t}$$
$$\ln\frac{3}{14} = -0.016754t$$
$$t = 91.94$$

About 92 more minutes

Ex. A pan of warm water (46°C) is put in the refrigerator. 10 minutes later, the water's temperature was 39°C. 10 minutes later, it was 33°C. Estimate the temperature of the refrigerator.

This problem is different because in both sentences, you are missing more than one piece of information.
You can still do the problem because it is SET UP for you to be able to do it

1st Sentence:

$T_0 = 46 \quad t = 10 \quad T = 39 \quad T_s = x$
$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$39 - X = (46 - x)e^{-10k}$$

2nd Sentence:

$T_0 = 39 \quad t = 10 \quad T = 33 \quad T_s = x$
$$T - T_s = (T_0 - T_s)e^{-kt}$$
$$33 - X = (39 - x)e^{-10k}$$

Look at the exponential term. The exponents are the same, so solve both for e^{-10k} and then substitute. You don't need to know what k is. It works because the cooling times are the <u>same</u>

$$e^{-10k} = \frac{39 - x}{46 - x}$$
$$e^{-10k} = \frac{33 - x}{39 - x}$$

$$\frac{39 - x}{46 - x} = \frac{33 - x}{39 - x}$$
$$(39 - x)(39 - x) = (46 - x)(33 - x)$$
$$1521 - 78x + x^2 = 1518 - 79x + x^2$$

$$1521 - 78x = 1518 - 79x$$
$$3 = x$$

The temp of the fridge is 3°C

Theorem: Resistance Proportional to Velocity

The velocity, V of a "coasting" object whose initial velocity is v_0 is given by:

$$v = v_0 e^{-(k/m)t}$$

where k is the rate of deceleration, m is the mass of the object, and t is the time since coasting began.

Proof:

- In Physics, the resistance encountered by a moving object (i.e. a car coasting to a stop) is proportional to the object's velocity.
  - The slower the object moves, the less resistance against the forward progress
- Mathematically:

Force = mass × acceleration

$$F = ma$$
$$F = mv'$$
$$F = m\frac{dv}{dt}$$

- Since the resistance force is proportion to the velocity, then

$$m\frac{dv}{dt} = -kv \qquad \int \frac{dv}{v} = \int -\frac{k}{m}dt \qquad v = e^{-\frac{k}{m}t+C} \qquad v_0 = Ke^{-\frac{k}{m}(0)}$$
$$\frac{dv}{dt} = -\frac{k}{m}v \qquad \ln|v| = -\frac{k}{m}t + C \qquad v = e^{-\frac{k}{m}t}\cdot e^C \qquad v_0 = K(1)$$
$$\frac{dv}{v} = -\frac{k}{m}dt \qquad v = e^{-\frac{k}{m}t+C} \qquad v = Ke^{-\frac{k}{m}t} \qquad v_0 = K$$

which when solved (just like we did earlier), results in:

$$v = v_0 e^{-(k/m)t}$$

ex. For a 50-kg ice skater, $k = 2.5$ kg/sec. Answer the following:

(a) How long will it take the skater to coast from 7 m/sec to 1 m/sec?

(b) How far will the skater coast before coming to a stop?

(a)

$$v = v_0 e^{-\left(\frac{k}{m}\right)t}$$
$$1 = 7e^{-\left(\frac{2.5}{50}\right)t}$$
$$\frac{1}{7} = e^{-0.05t}$$
$$\ln\frac{1}{7} = -0.05t$$
$$t = 38.918 \text{ seconds}$$

(b) recall $s(t) = \int v(t)\,dt$

$$s(t) = \int_0^\infty 7e^{-0.05t}dt$$
$$s(t) = \lim_{b\to\infty} 7\int_0^b e^{-0.05t}dt$$
$$s(t) = \lim_{b\to\infty} -140e^{-0.05}\Big|_0^b$$
$$s(t) = -140\lim_{b\to\infty}\left[\frac{1}{e^{0.05}} - e^0\right]$$
$$s(t) = -140(0-1) = 140$$

The skater will travel 140 m

Homework: Day 1: Pg. 357-358 #11(a), 12, 18 – 20, 26 – 28

Day 2: Pg. 357-360 #14-17, 21-24, 29, 32, 35, 36, 39, 42, 45, 46

6.5 Logistic Growth

Partial Fractions

ex. Evaluate: $\int \frac{5x-3}{x^2-2x-3}\,dx$

- There is no u-substitution that can be made for this
- No trig substitution as well.
- To evaluate this integral, we need to use the ***method of partial fractions***.
  - We will rewrite this fraction as the sum/difference of simpler fractions.
  - Factor the denominator: $(x-3)(x+1)$
  - Now rewrite:
    $$\frac{5x-3}{x^2-2x-3} = \frac{A}{x-3} + \frac{B}{x+1}$$
  - Combining the fractions on the right, we get:
    $$\frac{5x-3}{x^2-2x-3} = \frac{A(x+1)}{(x-3)(x+1)} + \frac{B(x-3)}{(x-3)(x+1)}$$
  - So: $5x-3 = A(x+1) + B(x-3) = Ax + A + Bx - 3B = (A+B)x + (A-3B)$
  - So we get the following system of equations:
    $$A+B=5$$
    $$A-3B=-3$$
    ➔ Solving the two equations you get $A = 3$, $B = 2$
  - So….
    $$\int \frac{5x-3}{x^2-2x-3}\,dx = \int \frac{3}{x-3} + \frac{2}{x+1}\,dx$$
    $$= 3\ln|x-3| + 2\ln|x+1| + C$$

To skip the step:
Multiply everything by the common denominator.

It will cancel the denominator on the left

For each fraction on the right, multiply the numerator by the factors that are not in the denominator of that fraction.

You can skip this step and go right to the other step by multiplying everything by the common denominator

Steps to Integrating Using Partial Fractions: $\int \frac{f(x)}{g(x)}\,dx$

- To use this method, the degree of the top must be less than degree of the bottom.
- You also need to be able to factor $g(x)$

1. Let $x-r$ be a linear factor of $g(x)$. Suppose $(x-r)^m$ is the highest power of $x-r$ that divides $g(x)$. To this factor assign the sum of the m partial fractions:
   $$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \frac{A_3}{(x-r)^3} + \ldots + \frac{A_m}{(x-r)^m}$$
   Do this for each distinct linear factor of $g(x)$.

2. Let x^2+px+q be the quadratic factor of g. Suppose $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. To this factor assign the sum of the n partial fractions:
   $$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \frac{B_3x+C_3}{(x^2+px+q)^3} + \ldots + \frac{B_nx+C_n}{(x^2+px+q)^n}$$
   Do this for each distinct quadratic factor of $g(x)$ that <u>cannot be factored</u> into linear factors with real coefficients.
3. Set the original fraction equal to the sum of all these partial fractions. Clean up the partial fractions into one fractions using common denominators and arrange the numerator in decreasing power of x.
4. Set the corresponding coefficients of the numerators equal to each other and solve the system of equations.
5. Rewrite the integral using the partial fractions (which should be able to be integrated) and find the solution.

Ex. Find $\int \frac{6x+7}{(x+2)^2}dx$

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$6x + 7 = A(x+2) + B$
$6x + 7 = Ax + (2A + B)$

$A = 6$
$2A + B = 7$
$12 + B = 7$
$B = -5$

$$\frac{6}{x+2} - \frac{5}{(x+2)^2}$$

$$\int \frac{6x+7}{(x+2)^2}dx = \int \frac{6}{x+2} - \frac{5}{(x+2)^2}dx$$

For both fractions, use:
$u = x + 2 \quad \rightarrow \quad du = dx$

$$6\int \frac{1}{u}\,du - 5\int u^{-2}du = 6\ln|u| + 5u^{-1}$$

$$\boxed{\int \frac{6x+7}{(x+2)^2}dx = 6\ln|x+2| + \frac{5}{x+2} + C}$$

Ex. $\int \frac{6x^2+10x+2}{x^3+3x^2+2x}dx$

$$\frac{6x^2+10x+2}{x^3+3x^2+2x} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$$
$6x^2 + 10x + 2 = A(x^2+3x+2) + B(x^2+2x) + C(x^2+x)$
$6x^2 + 10x + 2 = (A+B+C)x^2 + (3A+2B+C)x + 2A$

$A + B + C = 6$
$3A + 2B + C = 10$
$2A = 2$
$A = 1$
Substituting back:
$B + C = 5$
$2B + C = 7$
============
$B = 2 \quad \rightarrow \quad C = 3$

$$\frac{1}{x} + \frac{2}{x+1} + \frac{3}{x+2}$$

$$\int \frac{1}{x} + \frac{2}{x+1} + \frac{3}{x+2}dx$$

$$\boxed{\ln|x| + 2\ln|x+1| + 3\ln|x+2| + C}$$

Ex. $\int \frac{2x-4}{x^4-16}dx \quad x^4 - 16 = (x^2-4)(x^2+4) = (x+2)(x-2)(x^2+4)$

$$\frac{2x-4}{x^4-16} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{Cx+D}{x^2+4}$$
$2x - 4 = A(x^3 - 2x^2 + 4x - 8) + B(x^3 + 2x^2 + 4x + 8) + Cx^3 + Dx^2 - 4Cx - 4D$
$2x - 4 = (A+B+C)x^3 + (-2A+2B+D)x^2 + (4A+4B-4C)x + (-8A+8B-4D)$

1: $A + B + C = 0$
2: $-2A + 2B + D = 0$
3: $4A + 4B - 4C = 2$
4: $-8A + 8B - 4D = -4$
$4A + 4B - 4C = 2$
$-4A - 4B - 4C = 0$
$-8C = 2$
$C = -\frac{1}{4}$

$-8A + 8B - 4D = -4$
$8A - 8B - 4D = 0$
$-8D = -4$
$D = \frac{1}{2}$

$A + B - \frac{1}{4} = 0$
$-2A + 2B + \frac{1}{2} = 0$

$4A + 4B = 1$
$-4A + 4B = -2$

$8B = -1$
$B = -\frac{1}{8}$
$A = \frac{3}{8}$

$$\frac{3}{8}\int \frac{1}{x+2}dx - \frac{1}{8}\int \frac{1}{x-2}dx - \frac{1}{4}\int \frac{x-2}{x^2+4}dx$$

$$\boxed{\frac{3}{8}\ln|x+2| - \frac{1}{8}\ln|x-2| + \left[\frac{1}{2}\ln(x^2+4\} + \frac{1}{2}\tan^{-1}\frac{x}{2}\right] + C}$$

"Aside"
$$\int \frac{x-2}{x^2+4}dx$$
$$\int \frac{x}{x^2+4}dx - 2\int \frac{1}{x^2+4}dx$$
Let $u = x^2 + 4$
$du = 2xdx$
$\frac{1}{2}du = xdx$

$\frac{1}{2}\int \frac{1}{u}du$
$\frac{1}{2}\ln(x^2+4)$

Let $x = 2\tan\theta$
$dx = 2\sec^2\theta\, d\theta$
$\int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta + 4}$
$\frac{1}{2}\int d\theta$
$\frac{1}{2}\theta$
$\frac{1}{2}\tan^{-1}\frac{x}{2}$

Exponential Model

- As we have seen, population growth is an example of Exponential Growth

$$P = P_0 e^{kt}$$

where P_0 = original population (at t = 0)
t = time in ears
P = population after t years
k = **Relative Growth Rate:** $k = \dfrac{dP/dt}{P}$

- k can be determined from a population table. The rate would be the ratio of the current population to the preceding year's population

Year	Population (Millions)	Ratio
1986	4936	
		5023/4936 ≈ 1.0176
1987	5023	
		5111/5023 ≈ 1.0175
1988	5111	
		5201/5111 ≈ 1.0176
1989	5201	
		5329/5201 ≈ 1.0246
1990	5329	
		5423/5329 ≈ 1.0175
1991	5423	

Ex. Using the table above, estimate the world population (in millions) in the year 2010.

$$P = P_0 e^{kt} \quad \rightarrow \quad 5023 = 4936e^{k} \quad \rightarrow \quad k = \ln\left(\frac{5023}{4936}\right) = 0.017472$$

Based on this value of k: at t = 24 (year 2010):
$P = 4936e^{0.017472(24)} = 7507.369$

So, the population would be estimated at 7,507,369,000

The value of k is not constant, so we use it as an estimate

If you based k on the population change from 1986 to 1991, you would get

$$k = \frac{\ln\left(\frac{5423}{4936}\right)}{5} = 0.01881877$$

$$P = 4936e^{0.01881877(24)}$$

$$P = 7754.977$$

7,754,977 people

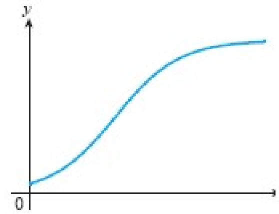
When using a table, there will be a slight error to take into consideration

The Logistic Differential Equation

- The exponential model assumes one thing: **unlimited growth**
  - This assumption is ok if the population is small
  - Eventually, the population will get to a point where it can not grow any further.
    - Therefore, the exponential model becomes no longer viable.
- A more realistic assumption is that the relative growth is positive
  - But decreases as the population increases
    - Due to:
      - Economic factors
      - Environmental factors

Consider a population, P, with a growth curve as a function of time.

- Suppose it begins increasing and concave up (i.e. exponential growth)
- Suppose it then turns increasing and concave down as it approaches the carrying capacity of its habitat.
- A graph of this situation is shown to the right.
  - It is called a <u>Logistic Curve</u>



- Since the start of the curve is exponential, then it can be modeled by the following differential equation:
  $$\frac{dP}{dt} = kP \quad \text{for some } k > 0$$
- The **<u>carrying capacity</u>**, M, is the maximum population that the environment is capable of sustaining life in the long run.
- If we want the growth rate to approach zero (growth to flatten out) as P approaches a maximal carrying capacity M, then we can introduce another factor to the differential equation:
  $$\frac{dP}{dt} = kP(M - P)$$
  - This is **<u>Logistic Differential Equation</u>**.

Note: If they give k relative to the with a population ratio $\frac{P}{M}$, then the formula could be: $\frac{dP}{dt} = \frac{k}{M}P(M - P)$

- Answering the Exploration 2 to the right:
  1. As P approaches 0 or M
  2. At the vertex, $P = \frac{m}{2}$
  3. $\frac{d^3P}{dt^2} = -2k$. Since $k > 0$, then $\frac{d^3P}{dt^2} < 0$, which makes it a maximum
  4. It is positive since $P > 0$ and $M - P > 0$ as well
  5. It would be negative because $M - P < 0$
  6. 0
  7.

EXPLORATION 2 Learning from the Differential Equation

Consider a (positive) population P that satisfies $dP/dt = kP(M - P)$, where k and M are positive constants.

1. For what values of P will the growth rate dP/dt be close to zero?
2. As a function of P, $y = kP(M - P)$ has a graph that is an upside-down parabola. What is the value of P at the vertex of that parabola?
3. Use the answer to part (2) to explain why the growth rate is maximized when the population reaches half the carrying capacity.
4. If the initial population is less than M, is the initial growth rate positive or negative?
5. If the initial population is greater than M, is the initial growth rate positive or negative?
6. If the initial population equals M, what is the initial growth rate?
7. What is $\lim_{t\to\infty} P(t)$? Does it depend on the initial population?

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$
$$1 = A(M-P) + BP$$
$$1 = AM + (B-A)P$$
$$AM = 1$$
$$B - A = 0$$
$$B = A$$
$$A = B = \frac{1}{M}$$

$$\frac{1}{P(M-P)}dP = kdt$$
$$\frac{1}{M}\int \frac{1}{P}dP + \frac{1}{M}\int \frac{1}{M-P}\,dP = \int k\,dt$$
$$\frac{1}{M}\ln P - \frac{1}{M}\ln|P-M| = kt + C$$
$$\frac{1}{M}\ln\frac{P}{P-M} = kt + C$$
$$\ln\frac{P}{P-M} = Mkt + C$$
$$\frac{P}{P-M} = Ce^{Mkt}$$
$$\frac{P-M}{P} = Ce^{-Mkt}$$
$$1 - \frac{M}{P} = Ce^{-Mlt}$$

$$\frac{M}{P} = 1 + Ce^{-Mkt}$$
$$P(t) = \frac{M}{1 + Ce^{-Mkt}}$$

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty}\frac{M}{1 + Ce^{-Mkt}} = \frac{M}{1+0} = M$$

So as $t\to\infty$, $P\to M$ *It does not depend on it*

- If P becomes higher than M, then the growth rate would be negative, and the population would be decreasing.

Ex. A national park is known to be capable of supporting no more than 100 grizzly bears. Ten bears are in the park at present. We model the population with a logistic differential equation with $k = 0.001$.

a. Find a logistic growth model $P(t)$ for the population.

b. When will the population reach 50?

(a) $\frac{dP}{dt} = kP(M-P)$

$$\frac{dP}{dt} = 0.001P(100-P)$$

$$\frac{1}{P(100-P)}dp = 0.001dt$$

$$\frac{1}{100}\int \frac{1}{P} + \frac{1}{100-P}dp = \int 0.001dt$$

$$\ln P - \ln|100-P| = 100(0.001t + C)$$

$$\ln\left(\frac{P}{100-P}\right) = 0.1t + C$$

$$\frac{P}{100-P} = e^{0.1t+C}$$

$$\frac{100-P}{P} = e^{-0.1t-C}$$

$$\frac{100}{P} - 1 = e^{-0.1t-C}$$

$$\frac{100}{P} = 1 + e^{-0.1t}e^{-c}$$

$$\frac{100}{P} = 1 + 9e^{-0.1t} \quad \rightarrow \quad P(t) = \frac{100}{1+9e^{-0.1t}}$$

"Aside"

$$\frac{1}{P(100-P)} = \frac{A}{P} + \frac{B}{100-P}$$

$$1 = A(100-P) + BP$$

$$1 = (B-A)P + 100A$$

$$A = \frac{1}{100}$$

$$B - A = 0$$

$$B = \frac{1}{100}$$

$$\frac{1}{100}\left[\frac{1}{P} + \frac{1}{100-P}\right]$$

Initial Condition

At $t = 0$, $P = 10$

$$\frac{100}{10} = 1 + e^{0}e^{-c}$$

$$10 = 1 + e^{-c}$$

$$c = -\ln 9$$

$$e^{-c} = 9$$

(b) $50 = \frac{100}{1+9e^{-0.1t}}$

$$\frac{1}{2} = \frac{1}{1+9e^{-0.1t}}$$

$$1 + 9e^{-0.1t} = 2$$

$$e^{-0.1t} = \frac{1}{9}$$

$$-0.1t = \ln\frac{1}{9}$$

$t = 21.97225 \approx 22$ years

The General Logistic Formula

The solution of the general Logistic Differential Equation

$$\frac{dP}{dt} = kP(M-P)$$

is

$$P(t) = \frac{M}{1+Ce^{-Mkt}}$$

where C is a constant determined by the initial condition. The carrying capacity M and the growth constant k are positive constants.

Homework: Day 1: Pg. 369-370 #1–7odd, 9–13, 15–21 odd, 25, 33, 35

Day 2: Pg. 369–371 #2-8even, 14–24,even, 29, 31, 34, 37, 39-44