Chapter 2 – Polynomial and Rational Functions

2.1 Complex Numbers

Def: The imaginary number *i* is defined as $i = \sqrt{-1}$ where $i^2 = -1$.

Principal Square Root of a Negative
$\sqrt{-n} = i\sqrt{n}$

Ex. Express using *i*:

1. $\sqrt{-25}$ 2. $\sqrt{-49}$ 3. $\sqrt{121i^2}$

Def: The set of all numbers of the form $a + bi$ with real numbers *a* and *b*, and imaginary unit *i*, is called the set of complex numbers. The *a* term is called the real term and the *bi* is called the imaginary term.

Ex. $-4 + 6i$ $2i = 0 + 2i$ $3 = 3 + 0i$

Property: Two complex numbers, $a + bi$ and $c + di$, are equal if and only if $a = c$ and $b = d$.

Addition/Subtraction of Complex Numbers:

1. $(a + bi) + (c + di) = (a + c) + (b + d)i$
2. $(a + bi) - (c + di) = (a - c) + (b - d)i$

Ex. Perform the indicated operation. Write the result as a complex number in standard form.

1. $(5 - 11i) + (7 + 4i)$ 2. $(-5 + i) - (-11 - 6i)$

3. $(5 - 2i) + (3 + 3i)$ 4. $(2 + 6i) - (12 - i)$

Multiplication of Complex Numbers

- Performed the same way as multiplying polynomials. Using….
  - The Distributive Property
  - FOIL

Ex. Find the product

1. $4i(3 - 5i)$ 2. $(7 - 3i)(-2 - 5i)$

3. $7i(2 - 9i)$ 4. $(7 - 2i)(-2 + 5i)$

Def: The complex conjugate of the number $a + bi$ is $a - bi$, and the complex conjugate of $a - bi$ is $a + bi$.

Property: The product of complex conjugates is $(a+bi)(a-bi)=a^2+b^2$.

Proof:

You can use this property to divide complex numbers

Ex. Divide and express result as a complex number in standard form:

1. $\frac{7+4i}{2-5i}$

2. $\frac{5+4i}{4-i}$

ex. Perform the indicated operation and write in standard form

1. $\sqrt{-18}-\sqrt{-8}$

2. $(-1+\sqrt{-5})^2$

3. $\frac{-25\ \ \sqrt{-5}}{15}$

Ex. Using the quadratic formula, solve the quadratic equation. Express the solution in standard form.

1. $3x^2-2x+4=0$

2. $x^2-2x+2=0$

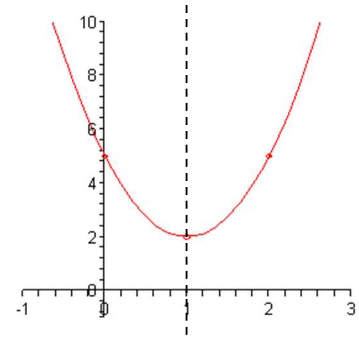
3. $x^2+x+1=0$

Homework: **Day 1: Pg. 284-285 #3-48 (3's), 51-60, 81-83**

Day 2: Pg. 284-285 #4-48(4's – omit multiples of 12), 61-62, 71, 72

Pg. 381 #1-12

2.2 Quadratic Functions



Def: A quadratic function is a function of the form $f(x) = ax^2 + bx +\ $, where *a, b,* and *c* are real numbers with $a \neq 0$.

- This is also a polynomial function whose greatest exponent is 2.

The graph of a quadratic function is a parabola. (see figure to the right).

- If $a > 0$ (positive), the parabola opens up.
- If $a < 0$ (negative), the parabola opens down.

Def: The vertex (or turning point) of the parabola is the lowest (or highest) point on the graph when it opens upward (or downward).

Def: For a parabolic function, the vertical line that passes through the vertex id called the *axis of symmetry*.

- It divides the parabola in half
- One side is the mirror image of the other.
- The equation of the axis of symmetry of the parabola whose equation is $f(x) = ax^2 + bx + c$, is

$$x = \frac{-b}{2a}$$

The Standard Form of a Quadratic Function

The quadratic function

$$f(x) = a(x-h)^2 + k\ , a \neq 0$$

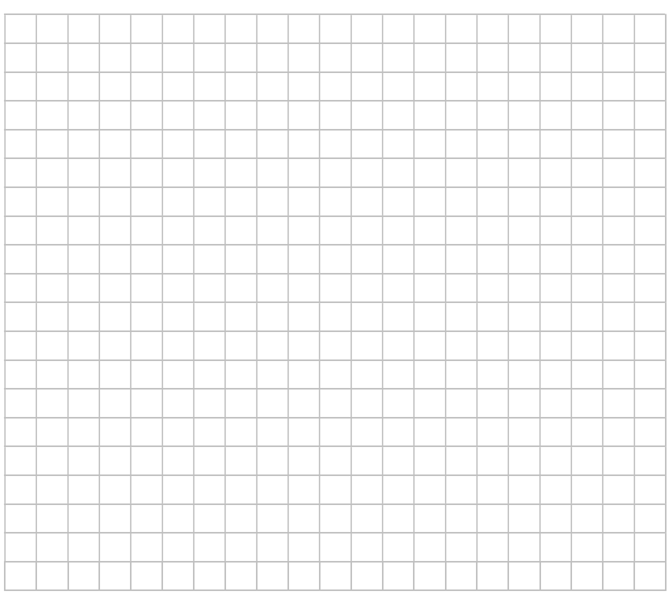
is in **Standard Form**. The vertex is at (*h*,*k*). The axis of symmetry is the line $x = h$. If *a* is positive, the parabola opens upward; if *a* is negative, the parabola opens downward.

To graph a Quadratic Function in Standard Form: $f(x) = a(x-h)^2 + k$

1. Determine which direction the parabola opens.
2. Plot the vertex of the parabola @ (*h*,*k*).
3. Find and plot any *x*-intercepts by solving $f(x) = 0$. The function's real zeros are the *x*-intercepts.
4. Find and plot the *y*-intercepts by finding *f*(0).
5. Smoothly connect the points.

Ex. Graph the quadratic function:

1. $f(x) = -2(x-3)^2 + 8$



2. $f(x) = -(x-1)^2 + 4$

To graph a Quadratic Function of the form $f(x) = ax^2 + bx + c$:

1. Find the axis of symmetry using the formula $x = \frac{-b}{2a}$. Sketch on the graph.
2. Plug the number found in (1) into the function to find the *y*-coordinate of the vertex. Plot this point.
3. Determine whether it opens upward (*a* positive) or downward (*a* negative).
4. From the axis of symmetry found in (1), make a table using 3 integer values less than and 3 integers greater than for x values and find the corresponding y values. Plot each of these points
5. Sketch the graph using a smooth curve through all of these points.

Ex. On the graph to the right, graph $f(x) = -x^2 - 2x + 1$

Ex. Consider the quadratic function.

(a) Determine, without graphing, whether the function has a minimum value or maximum value. Where does this value occur? What is the value?

(b) What is the domain and range for *f*?

1. $f(x) = -3x^2 + 6x - 13$

2. $f(x) = 4x^2 - 16x + 1000$

Homework: Pg. 298 #3-54 (3's)

2.2 Quadratic Functions (Day 2) - Applications

Ex. Studies have shown that when a football is punted, the nearest defensive player is 6 ft from the point of impact with the kicker's foot. The height of the punted football, *f*(*x*), in feet, can be modeled by the quadratic function $f(x) = -0.01x^2 + 1.18x + 2$, where *x* is the ball's horizontal distance, in feet, from the point of impact with the kicker's foot.

(a) What is the maximum height of the punt and how far from the point of impact does this occur?

(b) How far must the nearest defensive player, who is 6 ft from the kicker's point of impact, read to block the punt?

(c) If the ball is not blocked, how far down the field will it go before hitting the ground?

(d) Graph the football's path on the graph provided.

<u>How to Solve Problems involving Maximizing or Minimizing Quadratic Functions</u>:

1. Read the problem carefully and decide what quantity is to be max/minimized.
2. Use the condition of the problem to express the quantity as a function in one variable.
3. Write the function in the form: $f(x) = ax^2 + bx + c$
4. Calculate $-\frac{b}{2a}$. If *a* is negative, *f* will have a maximum value. If *a* is positive, *f* will have a minimum value. Plug in the number into *f* to get this value.
5. Make sure you answer the question asked.

Ex. Among all pairs of numbers whose difference is 10, find the pair of numbers whose product is as small as possible. What is this product?

Ex. Among all pairs of numbers whose difference is 8, find the pair of numbers whose product is as small as possible. What is this product?

Ex. You have 100 yards of fencing to enclose a rectangular region. What dimensions of the rectangle will give you the maximum enclosed area? What is this area?

Homework: Pg. 298-301 #1, 4, 5, 8, 10, 13, 14, 23, 25, 29, 31, 57, 61, 64-68, 71, 72

2.3 Polynomial Functions and Their Graphs

Def: Let n be a non-negative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers, with $a_n \neq 0$. The function defined by

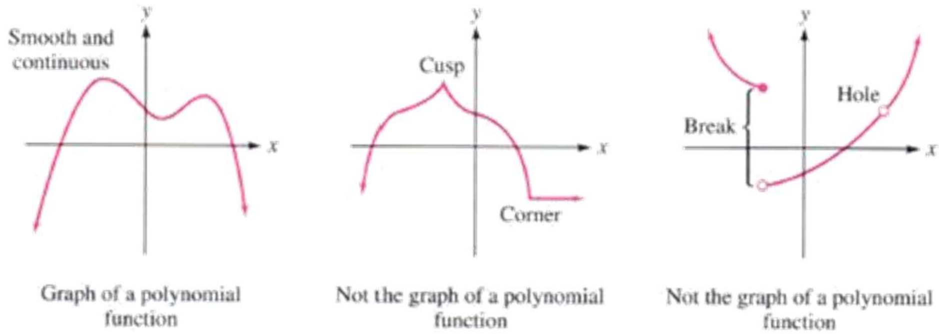
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

is called a **polynomial function of degree *n***. The number a_n is called the **leading coefficient** (the coefficient of the highest powered variable).

Ex. $f(x) = 4x^3 - 3x^2 - 5x + 6$ $\qquad$ $f(x) = -2x^4 - 5x + 1$

Note: Polynomial functions of degree greater than or equal to 2 are smooth and continuous curves.

- A *smooth* curve is a curve with no "sharp corners" – have rounded curves
- A continuous curve means no breaks in the curve – can be drawn without lifting the pencil off paper



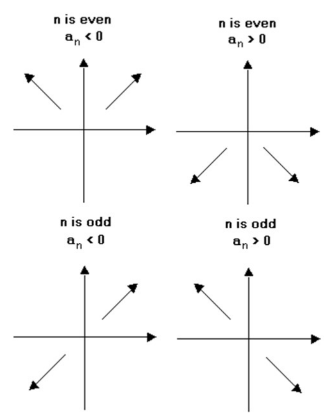
Def: The end behavior of a function is what a function is doing as x gets very large in a positive direction (right end-behavior) and in a negative direction (left end-behavior).

- The function is increasing or decreasing
- Is it approaching a particular y value (horizontal asymptote)

Property: The Leading Coefficient Test

For a polynomial function of degree n. The end-behavior of function will be

1. For even n: Both ends will increase or decrease based on the leading coefficient
   a. If leading coefficient is positive – both ends will rise
   b. If leading coefficient is negative – both ends will fall
2. For odd n: The end-behaviors will be in opposite directions
   a. If leading coefficient is positive: left falls, right rises
   b. If leading coefficient is negative: left rises, right falls



Ex. Describe the left and right end-behavior for the following functions

1. $f(x) = x^3 + 3x^2 - x - 3$ $\qquad$ 2. $f(x) = x^4 - x^2$ $\qquad$ 3. $f(x) = x - 3x^3$

Def: If f is a polynomial function, then the value(s) of x for which $f(x)$ is equal to 0 are called the **zeros** or **roots** of f.

- The zeros are also referred to as the solutions.
- Graphically: The zeros are where the function crosses the x-axis – the x-intercepts.

Ex. Find all the zeros of

1. $f(x) = x^3 + 3x^2 - x - 3$

2. $f(x) = x^3 + 2x^2 - 4x - 8$

3. $f(x) = -x^4 + 4x^3 - 4x^2$

4. $f(x) = x^4 - 4x^2$

<u>Theorem</u>: <u>The Intermediate Value Theorem</u>

Let *f* be a polynomial function with real coefficients. If *f*(*a*) and *f*(*b*) have opposite signs, then there exists *at least one value c* between *a* and *b* for which *f*(*c*) = 0 (in other words – there is at least one real root between *a* and *b*).

Ex. Show that $f(x) = x^3 - 2x - 5$ has a real zero between 2 and 3.

Ex. Show that $f(x) = 3x^3 - 10x + 9$ has a real zero between –3 and –2.

Homework: Pg. 312 #1–23odd, 33-40

2.4 Dividing Polynomials: Remainder and Factor Theorem

Recall: Long Division of Numbers:

Ex. $32567 \div 45$

Long Division of Polynomials:

1. Arrange the terms of both the dividend and divisor in decreasing order of powers
2. Divide the first term of the divisor into the first term of the dividend.
3. Multiply every term of the divisor by the expression found in step (2). Write beneath the dividend
4. Subtract the product from the dividend (change signs and add columns)
5. Bring down the next term of the dividend.
6. Repeat with the new expression.
7. Once you are out of terms to bring down, add the fraction consisting of the remainder over the divisor to the quotient

Division of Polynomials is done in a similar method:

Ex. Divide: $x^2 + 10x + 21$ by $x + 3$.

Ex. Divide $4 - 5x - x^2 + 6x^3$ by $3x - 2$

Ex. Divide $7 - 11x - 3x^2 + 2x^3$ by $x - 3$

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials, with $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exists unique polynomials $q(x)$ and $r(x)$ such that:

$$f(x) \quad = \quad d(x) \quad \cdot \quad q(x) \quad + \quad r(x)$$

The remainder, $r(x)$, equals 0 or it of degree less than the degree of $d(x)$. If $r(x) = 0$, we say $d(x)$ **divides evenly** into $f(x)$ and that $d(x)$ and $q(x)$ are **factors** of $f(x)$.

Ex. Divide $6x^4 + 5x^3 + 3x - 5$ by $3x^2 - 2x$

Ex. Divide $2x^4 + 3x^3 - 7x + 10$ by $x^2 - 2x$

<u>Synthetic Division</u>: A method of dividing a polynomial by $x - c$

1. Arrange the polynomial in decreasing powers, fill in missing terms with 0.
2. Solve $x - c = 0$ for x.
3. Write *c* as the divisor and to the right of it, write all the coefficients.
4. Write the leading coefficient on the bottom row (skip one row)
5. Multiply the number by c and place under the next coefficient.
6. Add the values in the column and write in the bottom row
7. Repeat steps (5) and (6) until you run out of coefficients.
8. The last number in the bottom row is the remainder.
9. The other numbers in the bottom row are the coefficients of a quotient polynomial.

Ex. Use synthetic division to divide

1. $5x^3 + 6x + 8$ by $x + 2$
2. $x^3 + 4x^2 - 5x + 5$ by $x - 3$
3. $x^3 - 7x - 6$ by $x + 2$

Homework: Pg. 324 #1-5odd, 17-31odd

2.5 Zeros of Polynomial Functions

Th: The Remainder Theorem: If a polynomial $f(x)$ is divided by $x - c$, the remainder is $f(c)$.

Ex. Use the Remainder Theorem to find $f(2)$

1. $f(x) = x^3 - 4x^2 + 5x + 3$
2. $f(x) = -3x^3 + 2x^2 - x + 1$

Th: The Factor Theorem: Let $f(x)$ be a polynomial.

(a) If $f(c) = 0$, then $x - c$ is a factor of $f(x)$.
(b) If $x - c$ is a factor of $f(x)$, then $f(c) = 0$.

Ex. If 3 is a zero of $f(x) = 2x^3 - 3x^2 - 11x + 6$, then find the other zeros.

Th: The Rational Zero Theorem

If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ has *integer* coefficients and $\frac{p}{q}$ (where $\frac{p}{q}$ is reduced to lowest terms) is a rational zero (root) of f, then p is a factor of a_0 (the constant) and q is a factor of the leading coefficient a_n.

- What this theorem does is it allows you to find all the possible roots of any polynomial using the roots of the constant term and the roots of the leading coefficient.,

Ex. Find all the possible roots of

1. $f(x) = x^4 - 3x + 8$
2. $3x^4 + 3x^3 - 2x + 6$
3. $f(x) = 15x^3 + 14x^2 - 3x - 2$

Ex. Find all the zeros for

1. $f(x) = x^3 + 2x^2 - 5x - 6$
2. $f(x) = x^3 + 7x^2 + 11x - 3$

Property: (a) If a polynomial equation is of degree *n*, then the number of roots counted separately is *n*.
(b) If $a + bi$ is a root of a polynomial with real coefficients ($b \neq 0$), then $a - bi$ is also a root.

Ex. Solve for *x*

1. $x^4 - 6x^2 - 8x + 24 = 0$

2. $x^4 - 6x^3 + 22x^2 - 30x + 13 = 0$

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients.

1. The number of ***positive real zeros*** of *f* is either:
   a. The same number of sign changes of *f*(*x*) *OR*
   b. Less than the number of sign changes of *f*(*x*) by a positive *even* integer. If *f* has only one variation in sign, then *f* has exactly ONE positive real zero.
2. The number of ***negative real zeros*** of *f* is either:
   a. The same number of sign changes of *f*(–*x*) *OR*
   b. Less than the number of sign changes of *f*(–*x*) by a positive *even* integer. If *f*(–*x*) has only one variation in sign, then *f* has exactly ONE negative real zero.

Ex. How many positive and negative real roots are there for

1. $f(x) = 3x^7 - 2x^5 - x^4 + 7x^2 + x - 3$

2. $f(x) = 4x^5 + 2x^4 - 3x^2 + x + 5$

3. $f(x) = -7x^6 - 5x^4 + x + 9$

4. $f(x) = x^3 + 2x^2 + 5x + 5$

Homework: Day 1: Pg. 335-336 #1–8, 9–15odd, 18-24even, 33-51 (3's)
Day 2: Pg. 336-339 #10-16even, 19, 23, 35, 37, 43, 47, 49, 72, 75-77, 87

2.6 Rational Functions and Their Graphs

Def: A rational function is a ratio of two polynomial functions.

$f(x) = \frac{p(x)}{q(x)}$, provided $q(x) \neq 0$

The domain of the a rational function is the set of all real numbers except the *x*-values where the denominator is equal to 0.

Ex. Find the domain of each of the following.

1. $f(x) = \frac{x^2+7x+9}{x(x-2)(x+5)}$
2. $f(x) = \frac{x^2-9}{x+3}$
3. $f(x) = \frac{x}{x^2+4}$

Def: The reciprocal function is the most basic of rational functions: $f(x) = \frac{1}{x}$.

What is the end behavior?

As *x* gets bigger and bigger, as we say "as x approaches infinity":____________________

As x approaches 0 from the left:____

As x approaches 0 from the right:____

As x gets more negative (bigger and bigger negative) "x approaches negative infinity":___________

In mathematics, the words "approaches" is written with the arrow symbol: →
Ex. "As x approaches infinity": $x \to \infty$

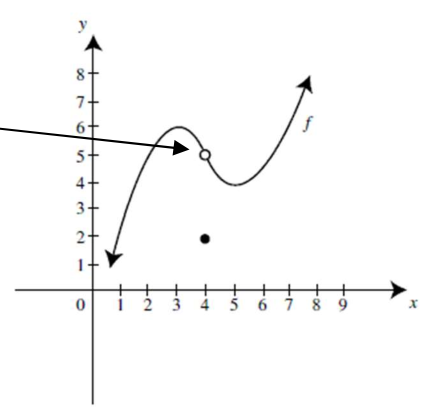
Arrow Notation:

1. $x \to a^+$ — *x* approaches *a* from the right
2. $x \to a^-$ — *x* approaches *a* from the left
3. $x \to \infty$ — *x* approaches infinity – *x* increases WITHOUT BOUND.
4. $x \to -\infty$ — *x* approaches negative infinity – *x* decreases WITHOUT BOUND.

*** In ***Calculus***, you use the concept called the ***limit*** to convey this idea of dealing with a functions behavior.

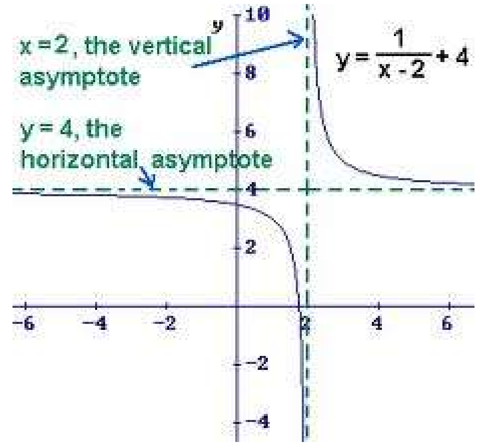
Def: The limit of a function, *f*(*x*), as *x* approaches *a*, written _____________, is L , the y value the function is approaching as $x \to a$.

In the diagram to the right, as $x \to 4$, the function approaches 5.



*** The curve does not have to exist at a point for the limit to exist there.

Asymptotes of Rational Functions



Def: The line *x* = *a* is a **vertical asymptote** of the graph of *f* if *f*(*x*) increases or decreases without bound as *x* approaches *a*.

Mathematically: *x* = *a* is a vertical asymptote if:

As $x \to a^-$: $f(x) = \pm\infty$

As $x \to a^+$: $f(x) = \pm\infty$

Def: The line *y* = *b* is a **horizontal asymptote** of the graph of *f* if *f*(*x*) approaches *b* as *x* increases or decreases without bound ($x \to \pm\infty$)

As $\to \pm\infty$: $f(x) \to b$

To find a vertical asymptote:

If $f(x) = \frac{p(x)}{q(x)}$ is a rational function in which *p*(*x*) and *q*(*x*) have NO common factors and *a* is a zero of *q*, the denominator, then *x* = *a* is a vertical asymptote of *f*.

Ex. Find the vertical asymptotes for each of the following, if any…..

1. $f(x) = \frac{x}{x^2-9}$ 2. $f(x) = \frac{x+3}{x^2-9}$ 3. $f(x) = \frac{x+3}{x^2+9}$

To find a horizontal asymptote: If $f(x) = \frac{p(x)}{q(x)}$ with 2 polynomial functions *p* (degree *m*) and *q* (degree *n*)

1. If *n* > *m*, then horizontal asymptote is the line *y* = 0 (the *x*-axis). (degree top < degree bottom)
2. If n < m, then there is no horizontal asymptote. (degree top > degree bottom)
3. If *n* = *m*, then the horizontal asymptote is $= \frac{a_m}{b_n}$, where a_m is the leading coefficient of *p* and b_n is the leading coefficient of *q*.

Ex. Find the horizontal asymptote, if any, of the graph of each rational function.

1. $f(x) = \frac{4x}{2x^2+1}$ 2. $f(x) = \frac{4x^2}{2x^2+1}$ 3. $f(x) = \frac{4x^3}{4x-21}$

4. $f(x) = \frac{3x^2-6x^3}{2x^3-1}$

Homework: Pg. 354 – 355 #1 – 35 odd

2.6 Rational Functions and Their Graphs (Day 2)

To Graph a Rational Function: $f(x) = \frac{p(x)}{q(x)}$ (no common factors for *p* and *q*)

1. Determine symmetry of *f*

   $f(-x) = f(x)$ - symmetry over *y*-axis

   $f(-x) = -f(x)$ - symmetry over origin (looks same rotated 180°)
2. Find the *y*-intercept (if any) – $f(0)$
3. Find *x*-intercept(s), if any, by solving $p(x) = 0$ (set top = 0 and solve)
4. Find any vertical asymptotes: set $q(x) = 0$ and solve for *x*
5. Find the horizontal asymptote, if any, by seeing what happens as $x \to \pm\infty$
6. Plot at least one point between and past each *x*-intercept and vertical asymptote
7. Sketch the function between and past the vertical asymptotes using information found.

Ex. On a set of axes, graph $f(x) = \frac{2x-1}{x-1}$

Ex. On a set of axes, graph $f(x) = \frac{3x^2}{x^2-4}$

Ex. On a set of axes, graph $f(x) = \frac{x^4}{x^2+1}$

Ex. On a set of axes, graph $f(x) = \frac{x-1}{x^2-4}$

A graph can have a "slant asymptote" or <u>oblique asymptote</u> if the degree of the top is ONE more than the degree of the bottom.

- To find the slant asymptote: Use long division – the quotient (no remainder) is the slant asymptote.

Ex. Find the slant asymptote and graph $f(x) = \frac{x^2-4x-5}{x-3}$

Homework: Pg. 355-356 #16-36even, 51-69(3's), 71, 73

2.7 Polynomial and Rational Inequalities

Def: A *polynomial inequality* is any inequality of the form

$$f(x) < 0,\ f(x) > 0,\ f(x) \le 0, \text{ or } f(x) \ge 0$$

where *f* is a polynomial function.

To solve a polynomial inequality:

1. Make sure the inequality is of defined form (make sure the one side is 0)
2. Solve the equation $f(x) = 0$. The values of x found are the boundary points(critical points).
3. Set up a number line and place the boundary points on it. This divides the number line into intervals.
4. Choose a value in each interval and test it in the polynomial (get its value).
   a. If the value is positive, then $f(x) > 0$ for this interval.
   b. If the value is negative, then $f(x) < 0$ for this interval.
5. Write the interval(s) for every section that satisfies the given inequality.

Ex. Solve and graph each of the following:

1. $x^2 - 5x - 6 < 0$

2. $x^2 - x - 20 \ge 0$

3. $2x^2 + x > 15$

4. $x^3 - 4x \ge 4 - x^2$

5. $x^3 + 4x^2 \le x + 3$

Def: A rational inequality is any inequality of the form

$$f(x) < 0,\ f(x) > 0,\ f(x) \le 0, \text{ or } f(x) \ge 0$$

where *f* is a rational function.

ex. $\frac{3x+3}{2x+4} > 0$

To solve a Rational Inequality:

1. Express the inequality so that one side is 0 and the other side is a single fraction.
2. Set the numerator and the denominator = 0 and solve for *x*
    a. Both will result in boundary points (but the denominator values can not be equal to 0)
3. Locate the boundary points on a number line (this gets intervals to test)
4. Choose an *x* value in each interval to test. (All you care about is if it is positive or negative)
5. Write the solution set, selecting the interval(s) that satisfy the inequality

    Note: Do not use ≥ or ≤ with any boundary point found from the denominator

Ex. Solve and graph the solution of each of the following:

1. $\frac{2x-4}{x+1} > 0$

2. $\frac{x+1}{x+3} \geq 2$

3. $\frac{x^2-9}{x+1} \leq 0$

Homework: Pg. 366-367 #3-57 (3's), 67, 69