Chapter 3 – Exponential and Logarithmic Functions
3.1 Exponential Functions

Def: The exponential function *f* with base *b* is defined by

$$y = f(x) = b^x$$

where *b* is a positive real number.

Ex. $f(x) = 2^x$ $\qquad$ $f(x) = 10^x$ $\qquad$ $f(x) = 2^{x-2}$ $\qquad$ $f(x) = \left(\frac{1}{4}\right)^x$

Note: To be an exponential function, the variable must be only in the exponent.

Ex. Evaluate $f(x) = 2^x$ when *x* = 2, 4, 1.5, and -3

Ex. Fill in the table and graph $f(x) = 2^x$ on the set of axes.

x	$f(x) = 2^x$
-3	
-2	
-1	
0	
1	
2	
3	

Ex. Fill in the table and graph $f(x) = \left(\frac{1}{2}\right)^x$ on the set of axes.

x	$f(x) = \left(\frac{1}{2}\right)^x$
-3	
-2	
-1	
0	
1	
2	
3	

<u>Characteristics of Exponential Functions ($y = f(x) = b^x$)</u>:

1. The domain is the set of all reals
2. The graphs always pass through (0,1). The y-intercept is 1 and there is no x-intercept.
3. If $b > 1$ then the graph is increasing throughout. The larger the value of b, the steeper the graph.
4. If $0 < b < 1$, then the graph is decreasing throughout. The smaller the value of b the steeper the decrease.
5. Exponential functions are one-to-one functions, therefore have inverses.
6. The x-axis is a horizontal asymptote.

<u>Transformations of Exponential Functions</u>: Let $f(x) = b^x$ and c be a positive real number

Transformation	Equation	Effect
Vertical translation (shift)	$g(x) = b^x + c$ $g(x) = b^x - c$	• Shifts the graph up c units • Shifts the graph down c units
Horizontal translation (shift)	$g(x) = b^{x+c}$ $g(x) = b^{x-c}$	• Shifts the graph left c units • Shifts the graph right c units
Reflection	$g(x) = -b^x$ $g(x) = b^{-x}$	• Reflects over x-axis • Reflects over y-axis
Vertical stretch/shrink (dilation)	$g(x) = cb^x$	• Vertically stretch ($c > 1$) • Vertical shrink ($0 < c < 1$)
Horizontal stretch/shrink (dilation)	$g(x) = b^{cx}$	• Horizontally stretch ($0 < c < 1$) • Horizontally shrink ($c > 1$)

Ex. Using the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^x - 3$

x	$f(x) = 2^x$
-2	
-1	
0	
1	
2	

Ex. Using the graph of $f(x) = 2^x$ to obtain the graph of $g(x) = 2^x + 2$

x	$f(x) = 2^x$
-2	
-1	
0	
1	
2	

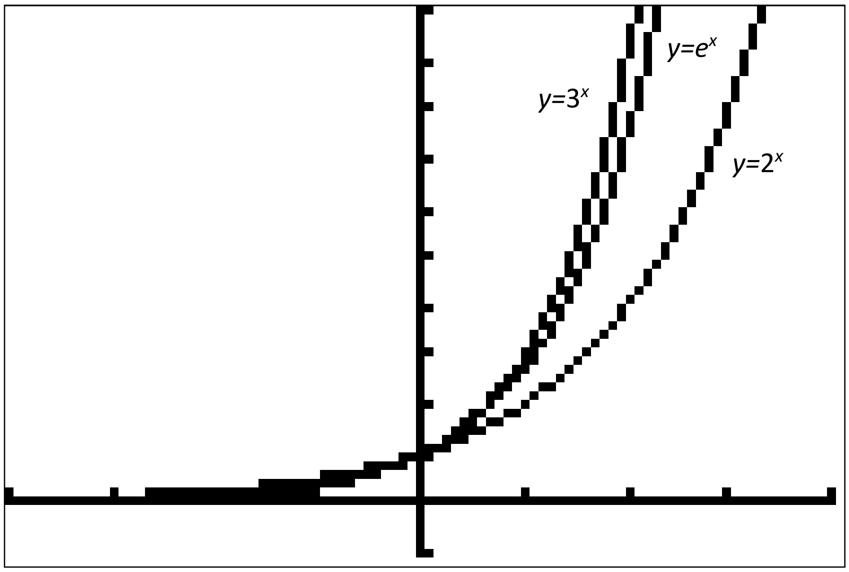
The Natural Base: ***e***

Def: The number *e* is an irrational number defined as the value of $(1+\frac{1}{n})^n$ for large *n*.

e = 2.71828182845904523……

Def: The Natural Exponential Function: $y = f(x) = e^x$

Looking at the graph:



The number *e* is very useful regarding growth and decay.

Continuously Compounded Interest:

The amount of money after *t* years, called *A*, with an investment of principle *P* at a rate of *r* is given by the formula:

$$A = Pe^{rt}$$

ex. Suppose \$20,000 is deposited in a money market account that pays at a rate of 5% per year compounded continuously. Determine the balance after 5 years. What if the interest was compounded quarterly?

Recall:
The amount of money after *t* years, called *A*, with an investment of principle *P* at a rate of *r* is compounded *n* times per year is:

$$A = P(1+\frac{r}{n})^{nt}$$

Homework: **Day 1:Pg. 396-397 #1-9odd, 11, 15, 19, 23, 29, 37, 39, 47, 53, 55**
Day 2: Pg. 396-398 #2-10even, 14, 20, 31, 32, 42, 44, 56,67, 69

3.2 Logarithmic Functions

Def: For $x > 0$ and $b > 0$ with $b \neq 1$,

$y = \log_b x$ is equivalent to $b^y = x$

The function $f(x) = \log_b x$ is the **logarithmic function with base *b***.

$y = \log_b x$ and $b^y = x$ mathematically mean the same thing.

- $b^y = x$ is in exponential form.
- $y = \log_b x$ is in logarithmic form.

ex. Write each of the following in exponential form.

1. $2 = \log_5 x$
2. $3 = \log_b 64$
3. $\log_4 22 = n$

ex. Write in logarithmic form

1. $15^3 = x$
2. $n^4 = 256$
3. $e^y = 15$

Note: The answer to a log statement is an exponent (the exponent of *b* that will give you *x*).

So..... $\log_4 64$ means: What is the exponent of 4 that gives you 64?

ex. Evaluate

1. $\log_{10} 100$
2. $\log_2 16$
3. $\log_5 \sqrt[3]{5}$
4. $\log_3 \frac{1}{27}$
5. $\log_4 2$

Properties of Logarithms:

1. $\log_n n = 1$
2. $\log_n 1 = 0$
3. $\log_n n^x = x$
4. $n^{\log_n x} = x$
5. $\log(ab) = \log a + \log b$
6. $\log \frac{a}{b} = \log a - \log b$
7. $\log_b x^n = n \log_b x$

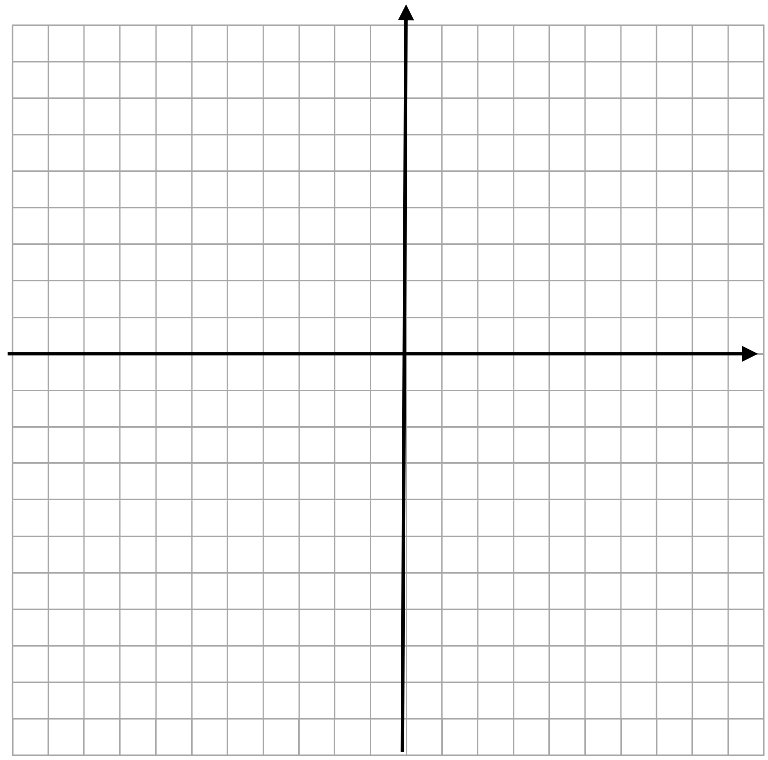
ex. Evaluate

1. $\log_4 4^5$
2. $6^{\log_6 8}$

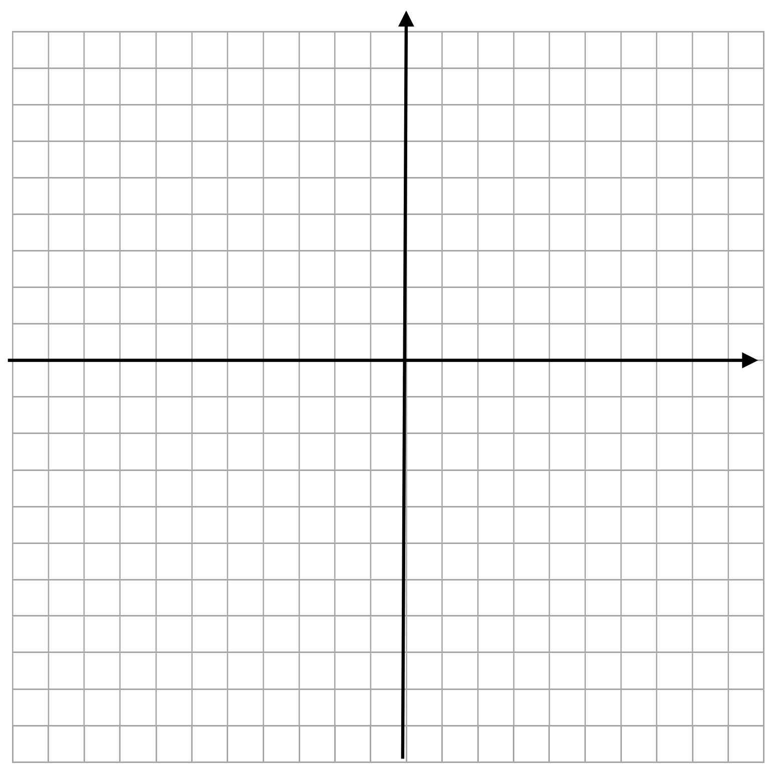
Graphing Logarithmic Functions:

- Logarithmic expressions are inverses of exponential functions.
  - This means that graph of $y = \log_b x$ will be the reflection of $y = b^x$ over the line *y* = *x*.
- So to graph a log graph, set up a table for the appropriate exponential and switch the coordinates.

ex. Graph $y = \log_2 x$



ex. Graph $y = \log_3 x$



Characteristics of Logarithmic Functions:

1. The domain of $f(x) = \log_b x$ consists of the set of all **positive real numbers: (0,∞)**
2. The graphs of all logarithmic functions, $f(x) = \log_b x$, pass through (1,0). Therefore the *x*-intercept is 1
   - There is no y-intercept
3. If the base *b* > 1, then $f(x) = \log_b x$ has a graph that increases as you move to the right.
4. If the base 0 < *b* < 1, then $f(x) = \log_b x$ has a graph that decreases as you move to the right.
5. The graph of $f(x) = \log_b x$ will approach the *y*-axis but will not touch or cross. The *y*-axis (*x* = 0) is a vertical asymptote.

Transformations of Logarithmic Functions: Let $f(x) = \log_b x$ and *c* be a positive real number

Transformation	Equation	Effect
Vertical translation (shift)	$g(x) = \log_b x + c$ $g(x) = \log_b x - c$	• Shifts the graph up *c* units • Shifts the graph down *c* units
Horizontal translation (shift)	$g(x) = \log_b(x + c)$ $g(x) = \log_b(x - c)$	• Shifts the graph left *c* units • Shifts the graph right *c* units
Reflection	$g(x) = -\log_b x$ $g(x) = \log_b(-x)$	• Reflects over *x*-axis • Reflects over *y*-axis
Vertical stretch/shrink (dilation)	$g(x) = c \log_b x$	• Vertically stretch (*c* > 1) • Vertical shrink (0 < *c* < 1)
Horizontal stretch/shrink (dilation)	$g(x) = \log_b(cx)$	• Horizontally stretch (0 < *c* < 1) • Horizontally shrink (*c* > 1)

Domain of Logarithmic Functions:

- Since a log is an exponential rewritten, and the base *b* > 0, then

  $y = \log_b x$ is equivalent to $b^y = x$

  Since *b* is positive, then a *x* must also be positive according to $b^y = x$
  - Therefore: The domain of $y = \log_b x$ is *x* > 0

  Therefore, if $y = \log_b f(x)$ where $f(x)$ is a function, then the domain is $f(x) > 0$, solved for *x*.

  ex. Find the domain of

  1. $f(x) = \log_4(x + 3)$

  2. $f(x) = \log_2(2x - 5)$

  3. $f(x) = \log_3(x^2 - 4)$

Def: A logarithm with base 10 is called a common logarithm (or common log). It is written $\log x$

Note: No base is written in a common log.

Def: A logarithm whose base is *e* is called a Natural Logarithm (or Natural log). It is written $\ln x$

$$\ln x = \log_e x$$

ex. Find the domain of each of the following:

1. $y = \ln(3 - x)$ 2. $y = \ln(x - 2)^2$ 3. $y = \ln x^2$

Homework: **Day 1: pg. 410 #1-41 odd**

Day 2: pg. 410-413 #4-100 (4's)

3.3 Properties of Logarithms

<u>Properties of Logs</u>:

1. Product Rule: $\log(mn) = \log m + \log n$
2. Quotient Rule: $\log\frac{m}{n} = \log m - \log n$
3. Power Rule: $\log x^n = n\log x$
4. Root Rule: $\log\sqrt[n]{x^p} = \frac{p}{n}\log x$
5. Reciprocal Rule: $\log\frac{1}{n} = -\log n$

<u>Properties of Exponents</u>:

1. Product Rule: $b^m \times b^n = b^{m+n}$
2. Quotient Rule: $\frac{b^m}{b^n} = b^{m-n}$
3. Power Rule: $(b^n)^p = b^{np}$
4. Root Rule: $\sqrt[n]{x^p} = x^{p/n}$
5. Negative Exponent Rule: $x^{-n} = \frac{1}{x^n}$

Ex. Use properties of logs to expand each of the following:

1. $\log_4(7 \cdot 5)$
2. $\log\ (10x)$
3. $\log_7\left(\frac{19}{x}\right)$
4. $\ln\left(\frac{e^3}{7}\right)$
5. $\log\ (7^4)$
6. $\ln\sqrt{x}$
7. $\log\left(x^2\sqrt{y}\right)$
8. $\ln\left(\frac{\sqrt[3]{x}}{36y^4}\right)$

Ex. Write as a single logarithm:

1. $\frac{1}{2}\log x + 4\log(x-1)$
2. $3\log(x+7) - \log x$
3. $4\ln x - 2\ln 6 - \frac{1}{2}\ln y$
4. $\frac{2}{3}\log x - 2\log 5 - 10\log y$

<u>Change of Base Property</u>:

For any logarithmic base b and any positive number m :

$$\log_b m = \frac{\ln m}{\ln b}$$

<u>Note</u>: You can use any log, not just ln i.e. $\log_b m = \frac{\mathrm{lo}}{\log b}$

Ex. Find the value of each of the following:

1. $\log_2 10$
2. $\log_5 140$
3. $\log e$

Homework: **Day 1: Pg. 421 #1-3, 6-102 (6's)**
Day 2: Pg. 421 #4-100 (4's not 12's)

3.4 Exponential and Logarithmic Equations

Def: An exponential equation is an equation containing a variable in an exponent.

Ex. $2^{3x-8} = 16$ $\quad 4^x = 15$

To solve an exponential equation: (Method 1: Expressing each side as a Power of the Same Base)

If $b^m = b^n$, then $m = n$

1. Rewrite the equation in the form $b^m = b^n$
2. Set $m = n$
3. Solve for the variable.

Ex. Solve

1. $2^{3x-8} = 16$
2. $27^{x+3} = 9^{x-1}$
3. $5^{3x-6} = 125$
4. $8^{x+2} = 4^{x-3}$

To solve an exponential equation: (Method 2: Using logarithms to solve)

1. Isolate the exponential expression.
2. Take the natural logarithm of both sides of the equation for bases other than 10. For base 10, use common logarithms.
3. Simplify using one of the following properties:
   $\ln b^x = x \ln b$ or $\ln e^x = x$ or $\log 10^x = x$
4. Solve for the variable.

Ex. Solve

1. $4^x = 15$
2. $10^x = 120{,}000$
3. $5^x = 134$
4. $40e^{0.6x} - 3 = 237$
5. $5^{x-2} = 4^{2x+3}$
6. $e^{2x} - 4e^x + 3 = 0$

Def: A logarithmic equation is an equation containing a variable in a logarithmic expression.

Ex. $\log_4(x+3)=2$ $\qquad\qquad$ $\ln(x+2)-\ln(4x+3)=\ln\left(\frac{1}{x}\right)$

To solve a Logarithmic Equation:

1. Express the equation in the form: $\log_b M = c$
2. Use the definition of a logarithm to rewrite the equation in exponential form:

   $\log_b M = c$ becomes $b^c = M$
3. Solve the exponential equation for the variable
4. Check the solution(s) in the original equation (make sure you only use values for which $M > 0$)

Ex. Solve

1. $\log_4(x+3)=2$

2. $3\ln(2x)=12$

3. $\log_2(x-4)=3$

4. $4\ln(3x)=8$

5. $\log_2 x+\log_2(x-7)=3$

6. $\log x+\log\,(x-3)=1$

7. $\ln\,(x+2)-\ln\,(4x+3)=\ln\left(\frac{1}{x}\right)$

8. $\ln(x-3)=\ln(7x-23)-\ln(x+1)$

Ex. Medical research indicates that the risk of having a car accident increases exponentially as the concentration of alcohol in the blood increases. The risk is modeled by

$$R = 6e^{12.77x}$$

where x is the blood alcohol concentration and R, given as a percent, is the risk of having a car accident. What blood alcohol concentration corresponds to a 17% risk of a car accident? What concentration corresponds to a 7% risk of a car accident? (this is the level in many states of losing your license if under 21)

Ex. The formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$ describes the accumulated value, A, of a sum of money, P, the principle, after *t* years at a annual percentage rate *r* (in decimal form) compounded *n* times in a year. How long will it take \$25,000 to grow to \$500,000 at 9% annual interest compounded monthly? Daily?

Homework: Pg. 432-433 #1-9odd, 12-22even, 24-48 (3's), 49, 53, 57, 66, 69, 75, 79, 85, 88

3.5 Exponential Growth and Decay

<u>Exponential Growth/Decay Models</u>:

The mathematical model for <u>exponential growth</u> or <u>decay</u> is given by:

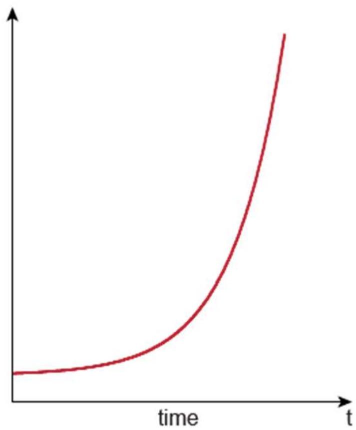
$$y = y_0 e^{kt} \quad \text{or} \quad f(t) = y_0 e^{kt}$$

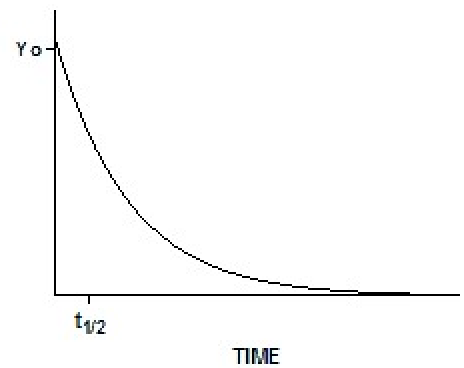
where y_0 (pronounced "y-naught") is the initial amount of something, *t* is time (in some unit), and *k* is a constant representing growth/decay rate.

If $k > 0$, then the model represents GROWTH (an increase in size)

If $k < 0$, then the model represents DECAY (a reduction in size)

Graphically: Growth Decay





The *horizontal*-axis (or *t*-axis) always represents time while the *vertical*-axis always represents "population" or the number of items in the model.

Ex. The US population, in 1970 was 203.3 million people. In 2007, the population had grown to 300.9 million people.

(a) Write an exponential growth rate that models this.

(b) According to the model, what year would the population be 315 million people?

(c) According to the model, what would the population be in 2013?

Ex. In 1990, the population of Africa was 643 million and by 2006 it was 906 million. What year would you expect the population to reach 2 billion (2,000 million) according to the model?

Ex. Carbon-Dating is used to determine how old fossils, artifacts, and even trees. Carbon-14 decays exponentially with a *half-life* (the time it takes for half a sample to disintegrate) of 5,715 years. Using this fact

(a) Find an exponential decay model for Carbon-14 decaying (leave in terms of y_0).

(b) In 1947, the Dead Sea Scrolls were discovered. Analysis indicated that the scroll wrappings contained 76% of their original Carbon-14. Estimate the age of the Dead Sea Scrolls (in 1947).

Ex. Strontium-90 is a waste product from nuclear reactors. As a consequent of nuclear fallout from nuclear testing, we have a measureable amount of Strontium-90 in our bones. If the half-life of Strontium-90 is 28 years, then find the amount of time it would take 60 grams of Strontium-90 to decay to 10 grams. How long would it take a sample to decay to 10% of its original amount?

Ex. The function $f(t) = \frac{30,000}{1+20e^{-1.5t}}$ describes the number of people, $f(t)$, who have become ill with the flu t weeks after its initial outbreak in a town of 30,000 people.

(a) How many people became ill with the flu when the epidemic began?

(b) How many people were ill by the end of the 4th week?

(c) Approximately how many weeks will it take for 5,000 people to become ill?

<u>Newton's Law of Cooling</u>:

The temperature, *T*, of a heated object at time *t* is given by:

$$T = T_S + (T_0 - T_S)e^{-k}$$

where T_S is the surrounding temperature (constant), T_0 is the original temperature of the object, and *k* is a constant associated with the cooling object.

Ex. A pie is removed from the oven has an internal temperature of 210°F. It is left to cool in a room with a constant temperature of 70°F. After 30 minutes, the temperature of the pie is 140°F.

(a) Write a model for this situation using Newton's Law of Cooling

(b) What is the temperature of the pie after 40 minutes?

(c) When will the temperature of the pie be 90°F?

Newton's Law of Cooling can also be used for heating (use a positive value of *k*)

Ex. A frozen steak initially has a temperature of 25°F is left to thaw in a room whose temperature is 70°F. After 10 minutes, the temperature of the steak has risen to 32°F.

(a) What will the steak's temperature be in another 10 minutes?

(b) When will the steak's temperature be 50°F (based on it originally being taken out of the freezer)?

Homework: **Day 1: Pg. 447-449 #1-6, 8, 15-17, 28, 32, 33, 35, 43, 45, 47, 50**

Day 2: Pg. 447-451 #7, 9-14, 18-20, 21-29odd, 30, 31, 34, 44, 46, 49, 83