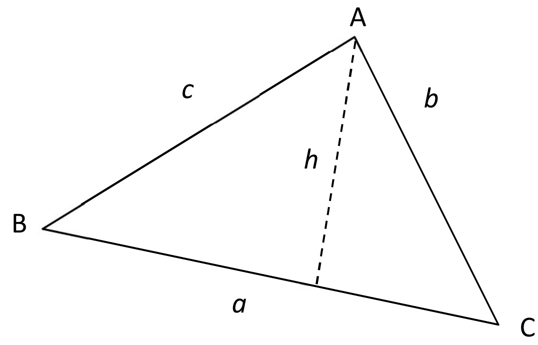
**Chapter 6 – Additional Topics in Trigonometry**
**6.1 The Law of Sines**

An *oblique triangle* is a triangle that does not contain a right angle.

Given $\Delta ABC$ as shown to the right:




Let $h$ be the length of the height drawn from $A$ to $BC$

This forms two right triangles, using right triangle trig, we get

$$\sin B = \tfrac{h}{c}$$
$$\sin C = \tfrac{h}{b}$$

these lead to the following equations after cross multiplying:

$$h = c\sin B$$
$$h = b\sin C$$

Since h is equal to both of these then

$$b\sin C = c\sin B$$
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly, we can prove the same with side $a$ and angle $A$, this leads to **<u>The Law of Sines</u>**:

**<u>Theorem</u>: The Law of Sines**

If $a$, $b$, and $c$ are the lengths of the sides of a triangle and $A$, $B$, and $C$ are the angles opposite these sides, then

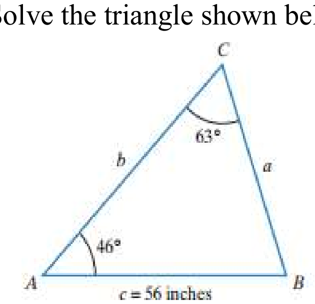
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The ratio of the length of the side of any triangle to the sine of the angle opposite that side is the same for all sides of the triangle.

This law can be used to find the remaining parts of an oblique triangle when we know either of the following:

1. two sides and an angle opposite one of the sides (SSA)
2. two angles and any side (AAS or ASA)

Ex. Solve the triangle shown below with $A = 46°$, $C = 63°$, and $c = 56$ inches. Round lengths to nearest tenth.



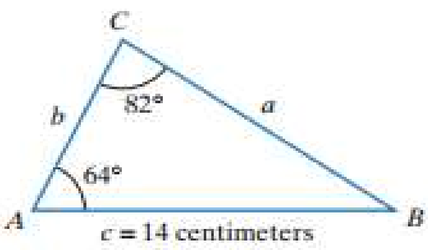

$m < B = 180 - 63 - 46 = 71$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$
$$\frac{a}{\sin 46} = \frac{56}{\sin 63}$$
$$a = \frac{56\sin 46}{\sin 63}$$
$$a = 45.2\ in$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$
$$\frac{b}{\sin 71} = \frac{56}{\sin 63}$$
$$b = \frac{56\sin 71}{\sin 63}$$
$$b = 59.4\ in$$

Demonstrate the same results using $\frac{a}{\sin A} = \frac{b}{\sin B}$ once you have found $a$. Verify by using longer side opposite bigger angles!

Ex. Solve the triangle shown below with $A = 64°$, $C = 82°$, and $c = 14$ cm. Round lengths to the nearest tenth.




$m < B = 180 - 64 - 82 = 34$

$\frac{a}{\sin} = \frac{c}{\sin C}$

$\frac{a}{\sin 64} = \frac{14}{\sin 82}$

$a = \frac{14 \sin 64}{\sin 82}$

$a = 12.7\ cm$

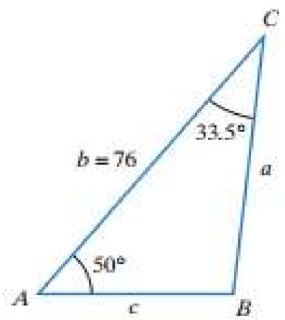
$\frac{b}{\sin B} = \frac{c}{\sin C}$

$\frac{b}{\sin 34} = \frac{14}{\sin 82}$

$b = \frac{14 \sin 34}{\sin 82}$

$b = 7.9\ cm$

Ex. In $\Delta ABC$, shown below, with $A = 50°, C = 33.5°$, and $b = 76$. Solve the rest of the triangle. Round lengths to the nearest tenth.




$m < B = 180 - 33.5 - 50 = 96.5$

$\frac{b}{\sin b} = \frac{c}{\sin C}$

$\frac{76}{\sin 96.5} = \frac{c}{\sin 33.5}$

$c = \frac{76 \sin 33.5}{\sin 96.5}$

$c = 42.2$

$\frac{b}{\sin B} = \frac{a}{\sin A}$

$\frac{76}{\sin 96.5} = \frac{a}{\sin 50}$

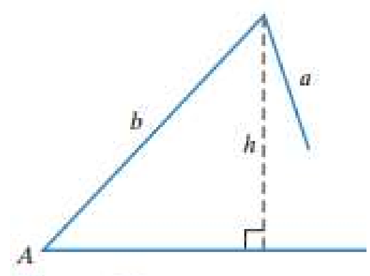
$b = \frac{76 \sin 50}{\sin 96.5}$

$a = 58.6$

**<u>Application</u>: The Ambiguous Case**

Consider a triangle in which *a, b,* and *A* are given. You can determine the number of possible triangles doing the following:

1. Calculate the height $h$, using $h = b \sin A$
2. Test the height vs. side $a$ to determine the # of triangles:
   - (i) If $a < h$: 0 triangles
   - (ii) If $a = h$: 1 triangle (right triangle)
   - (iii) If $h < a < b$: 2 distinct triangles
   - (iv) If $a > h$ and $a > b$, then 1 triangle




Ex. Determine the number of possible triangles given the following: Solve the rest of the triangle.

1. $A = 43°, a = 81,$ and $b = 62$

   $h = b \sin A = 62 \sin 43 = 42.2839$

   *Since a > h and a > b: 1 triangle*

2. $A = 57°, a = 33,$ and $b = 26$

   $h = 26 \sin 57 = 21.8054$

   *Since h < a and a > b: 1 triangle*

3. $A = 40°, a = 54,$ and $b = 62$

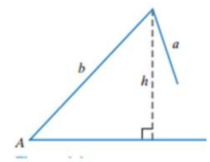
$h = b \sin A = 62 \sin 40 = 39.85283$

*Since $h < a < b$: 2 triangles*

**Application: The Ambiguous Case**

Consider a triangle in which $a$, $b$, and $A$ are given. You can determine the number of possible triangles doing the following:

1. Calculate the height $h$, using $h = b \sin A$
2. Test the height vs. side $a$ to determine the # of triangles:
   (i) If $a < h$: 0 triangles
   (ii) If $a = h$: 1 triangle (right triangle)
   (iii) If $h < a < b$: 2 distinct triangles
   (iv) If $a > h$ and $a > b$, then 1 triangle



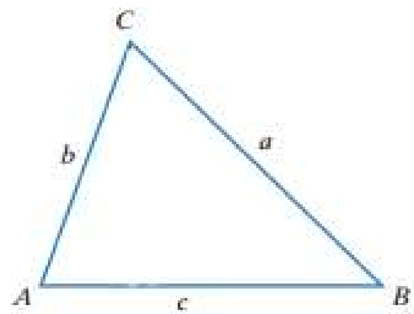

Perhaps demonstrate a right triangle example:
$A = 30$, $b = 12$, and $a = 6$

**Th: The Area of an Oblique Triangle**

Given $\Delta ABC$ with known sides $a$ and $b$, and included angle $C$, the area of the triangle is

$$Area = \frac{1}{2}\, ab \sin C$$

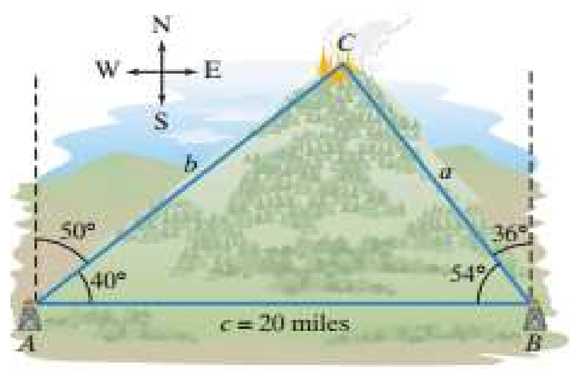
Likewise, $Area = \frac{1}{2}\, ac \sin B$ and $Area = \frac{1}{2}\, bc \sin A$




Ex. Find the area of the triangle have two sides of length 10 and 24 meters and an included angle of 62°

$$\text{Area} = \frac{1}{2} ab \sin C = \frac{1}{2}(10)(24) \sin 62 = 105.9537111 \approx 105.95 \text{ m}^2$$

Ex. Two fire-lookout stations are 20 miles apart, with Station B directly East of Station A. Both stations spot a fire on a mountain to the north. The bearing from Station A to the fire is $N50°E$ (which means 50° east of north). The bearing from Station B to the fire is $N36°W$ (36° west of North). How far, to the nearest tenth of a mile, is the fire from Station A?
(A map is shown to the right)




*Using the bearing to get angles A and B:*

$m < A = 40$ and $m < B = 54$ $\therefore m < C = 86$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 40} = \frac{20}{\sin 86}$$

$$a = \frac{20 \sin 40}{\sin 86} = 12.88714463 \approx 12.9 \text{ miles}$$

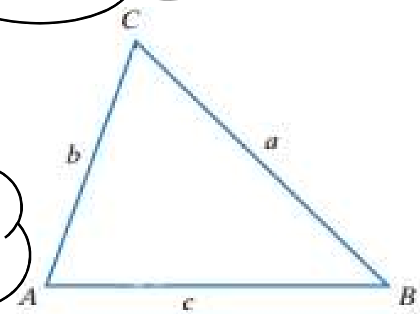
**Homework: Day 1: pg. 651-655 #1-4, 10, 15, 18-30(3's), 34, 38, 39**
**Day 2: pg. 651-655 #5, 14, 20, 33, 35, 37, 42, 50, 52, 57**

## 6.2 The Law of Cosines

When the problem involves SSSA (3 sides and angle) use this

**Th: The Law of Cosines**

If *A, B,* and *C* are the measures of the angles of a triangle, and *a, b,* and *c* are the lengths of the sides opposite these angles, then



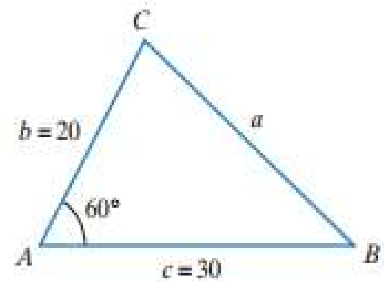

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Here is another version of the formula

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

The square of a side of a triangle equals the sum of the squares of the other two sides minus twice their product times the cosine of their included angles.

Ex. Given $\Delta ABC$ shown to the right, with $A = 60°, b = 20,$ and $c = 30$. Find the missing parts of the triangle. Round lengths to the nearest tenth and angle measures to the nearest degree.




*Since you are given SAS and looking for the third side – use Law of Cosines:*

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 20^2 + 30^2 - 2(20)(30)\cos 60$

$a^2 = 400 + 900 - 1200\left(\frac{1}{2}\right)$

$a^2 = 1300 - 600$

$a^2 = 700$

$a = \sqrt{700} = 26.45751311 \ \approx 26.5$

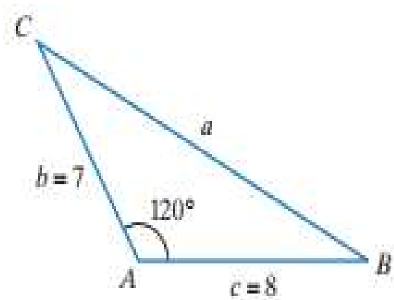
*To get angles, find first angle using the Law of Sines, and then find the 3rd angle:*

$\frac{a}{\sin A} = \frac{b}{\sin B}$

$\frac{26.457513}{\sin 60} = \frac{20}{\sin B}$ $\qquad B = \sin^{-1}\left(\frac{20\sin 60}{26.457513}\right) = 40.89339465 \ \approx \ 41°$

$\sin B = \frac{20\sin 60}{26.457513}$ $\qquad m < C = 180 - 60 - 61 = 79°$

Ex. Given $\Delta ABC$ shown to the right, with $A = 120°, b = 7,$ and $c = 8$. Find the missing parts of the triangle. Round lengths to the nearest tenth and angle measures to the nearest degree.




*Since you are given SAS and looking for the third side – use Law of Cosines:*

$a^2 = b^2 + c^2 - 2bc\cos A$

$a^2 = 7^2 + 8^2 - 2(7)(8)\cos 120$

$a^2 = 49 + 64 - 112\left(-\frac{1}{2}\right)$

$a^2 = 113 + 56$

$a^2 = 169$

$a = \sqrt{169} = 13$

*To get angles, find first angle using the Law of Sines, and then find the 3rd angle:*

$\frac{a}{\sin A} = \frac{b}{\sin B}$ $\qquad B = \sin^{-1}\left(\frac{7\sin 120}{13}\right) = 27.7957725 \ \approx \ 28°$

$\frac{13}{\sin 120} = \frac{7}{\sin B}$ $\qquad m < C = 180 - 120 - 28 = 32°$

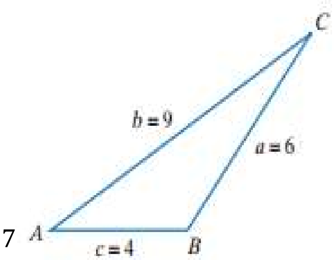
$\sin B = \frac{7\sin 120}{13}$

Ex. Solve triangle ABC if $a = 6, b = 9,$ and $c = 4$. Round angle measures to the nearest degree.

*Whenever you have SSS and looking for an angle or angles – use LAW OF COSINES*

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$
$$\cos C = \frac{6^2 + 9^2 - 4^2}{2(6)(9)}$$
$$\cos C = \frac{36 + 81 - 16}{108}$$
$$C = \cos^{-1}\left(\frac{101}{108}\right) = 20.74191648 \approx 21$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos C = \frac{6^2 + 4^2 - 9^2}{2(6)(4)}$$
$$\cos C = \frac{36 + 16 - 81}{48}$$
$$C = \cos^{-1}\left(\frac{-29}{48}\right) = 127.1688997 \approx 127$$




$$m < A = 180 - 21 - 127 = 32$$

You could have used the Law of Sines of find $< B$ or $< A$, then find the other angle

**<u>Th</u>: Heron's Formula for the Area of a Triangle**

The area of a triangle with sides *a, b,* and *c* is

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

Where *s* is one-half the perimeter (called the *semi-perimeter*): $s = \frac{1}{2}(a + b + c)$

Ex. Find the area of the triangle whose sides are 12 yds, 16 yds, and 24 yds. Round the area to the nearest tenth.

$$s = \tfrac{1}{2}(12 + 16 + 24) = 26$$
$$Area = \sqrt{26(26-12)(26-16)(26-24)}$$
$$Area = \sqrt{26(14)(10)(2)}$$
$$Area = \sqrt{7280} = 85.322916 \approx 85.3 \text{ yds}^2$$

*Using the Law of Cosines to find an angle and the area formula:*

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{12^2 + 16^2 - 24^2}{2(12)(16)}$$
$$\cos C = \frac{144 + 256 - 576}{384}$$
$$C = \cos^{-1}\left(-\frac{176}{384}\right) = 117.279613 \approx 117.2796$$

$$Area = \frac{1}{2}ab\sin C = \frac{1}{2}(12)(16)\sin 117.2796 = 85.3229$$

Ex. Find the area of the triangle whose sides are 6 m, 16 m, and 18 m. Round to the nearest tenth of a square meter. Also, find the measure of the smallest angle of the triangle.

$$s = \tfrac{1}{2}(6 + 16 + 18) = 20$$
$$Area = \sqrt{20(20-6)(20-16)(20-18)}$$
$$Area = \sqrt{20(14)(4)(2)}$$
$$Area = \sqrt{2240} = 47.3286383 \approx 47.3 \text{ m}^2$$

*Using the Law of Cosines to find an angle (remember smallest angle opposite shortest side):*

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{18^2 + 16^2 - 6^2}{2(18)(16)} = \frac{324 + 256 - 36}{576}$$

$$C = \cos^{-1}\left(\frac{544}{576}\right) = 19.18813645 \approx 19.2°$$

**Homework: pg. 661-663 #1-7odd, 9-30 (3's), 31, 33, 41, 43-44, 49**

## 6.3 Polar Coordinates

**<u>recall</u>:** The *<u>rectangular coordinate system</u>* is a method of graphing that assigns a set of ordered pairs, called *<u>coordinates</u>* for every point in the plane.

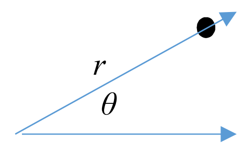
- the ordered pair are designated $(x, y)$

*There is another way of assigning points in the plane that uses trigonometry. It is called **<u>Polar Coordinates</u>***

**<u>Def</u>: The Polar Coordinate System**

- Instead of a set of axes, the Polar Coordinate System is based on a one ray axis that starts at the origin (called the ***<u>pole</u>***) at extends to the right.
  - This is designated as 0° or 0 radians
- Points are plotted using a set of coordinates $(r, \theta)$, where $r$ is the directed distance from the pole and $\Theta$ is the measure of the angle from the axis (+ is counter clockwise, – is clockwise) .

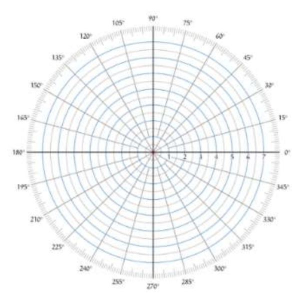
Example: (4,30°) would place a point 4 units from the pole and 30° in a counter-clockwise direction from the 0 axis.

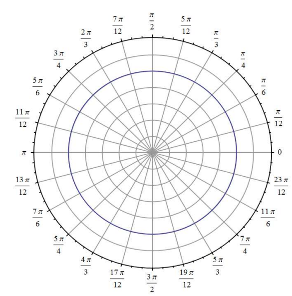



*Note: r can be negative*

$(r, \theta)$ are called *<u>Polar Coordinates</u>*

As with Rectangular Coordinates, Polar Coordinates has a specific graphing set up:

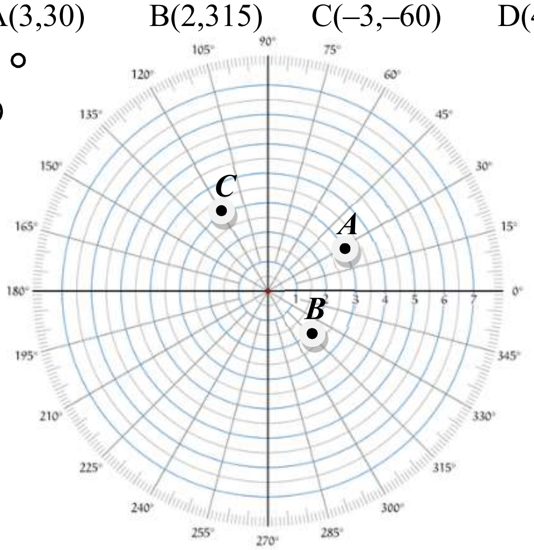




Ex. On the graph paper below, plot the following points: $(r, \theta)$

A(3,30) B(2,315) C(–3,–60) D(4, 180) E(2.5, 210) F(0,–120)

Do first 3 and have students do next 3




**The Sign of *r* and where to plot**

The point $P(r, \theta)$ is located $|r|$ units from the pole. If $r > 0$, then the point lies on the terminal side of $\Theta$. If $r < 0$, then the points lies along the ray opposite the terminal side.

If $r = 0$, then place the point at the pole.

**Property**: If $n$ is any integer, the point $(r,\theta)$ can be represent as:

$$(r, \theta) = (r, \theta + 2n\pi) \text{ or } (r, \theta) = (-r, \theta + \pi + 2n\pi)$$

Ex. Find another representation of the point $\left(2, \frac{\pi}{3}\right)$

(d) $r$ is positive and $2\pi < \theta < 4\pi$
(e) $r$ is negative and $0 < \theta < 2\pi$
(f) $r$ is positive and $-2\pi < \theta < 0$

*(a) Add* $2\pi$: $\left(2, \frac{\pi}{3} + 2\pi\right) = \left(2, \frac{7\pi}{3}\right)$
*(b) Change r & Add* $\pi$: $\left(-2, \frac{\pi}{3} + \pi\right) = \left(-2, \frac{4\pi}{3}\right)$ (from property with $n = 0$)
*(c) Subtract* $2\pi$: $\left(2, \frac{\pi}{3} - 2\pi\right) = \left(2, -\frac{5\pi}{3}\right)$

**Property: Relationship between Polar and Rectangular Coordinates**

$(x, y)$ and $(r, \theta)$ are the same point if

$$x = r\cos\theta \qquad x^2 + y^2 = r^2$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

Ex. Find the rectangular coordinates with the following polar coordinates:

a. $\left(2, \frac{3\pi}{2}\right)$ b. $\left(-8, \frac{\pi}{3}\right)$ c. $(3, \pi)$ *answer:* $(-3,0)$

a.
$x = r\cos\theta = 2\cos\frac{3\pi}{2}$
$x = 2(0) = 0$

$y = r\sin\theta = 2\sin\frac{3\pi}{2}$
$y = 2(-1) = -2$

$(0,-2)$

b.
$x = r\cos\theta = -8\cos\frac{\pi}{3}$
$x = -8\left(\frac{1}{2}\right) = -4$

$y = r\sin\theta = -8\sin\frac{\pi}{3}$
$y = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$

$(-4, -4\sqrt{3})$

Ex. Express each of the following rectangular coordinates as polar coordinates

a. $(-1, \sqrt{3})$

$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + \sqrt{3}^2} = \sqrt{1+3} = \sqrt{4}$
$r = 2$
QUAD II (look at point coordinates)
$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3})$
Ref angle =60 → Quad 2 : 120 $\frac{2\pi}{3}$ $\left(2, \frac{2\pi}{3}\right)$

b. $(3, -\sqrt{3})$

$r = \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-\sqrt{3})^2} = \sqrt{9+3} = \sqrt{12}$
QUAD IV (look at point coordinates)
$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$
Ref angle =30 → Quad 4 : 330 $\frac{11\pi}{6}$ $\left(\sqrt{12}, \frac{11\pi}{6}\right)$

c. $(4,3)$

$r = \sqrt{x^2 + y^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$
QUAD I (look at point coordinates)
$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{3}{4}\right) = 0.6435\ rad$ = 36.870°
Quad 1 : $(5, 36.870°)$

d. $(0, -4)$

$r = \sqrt{x^2 + y^2} = \sqrt{(0)^2 + (-4)^2} = \sqrt{16} = 4$
On negative $y$-axis: 270° or $\frac{3\pi}{2}$

$\left(4, \frac{3\pi}{2}\right)$

**Def:** A polar equation is an equation whose variables are $r$ and θ.

Ex. $r = \frac{5}{\cos\theta + \sin\theta}$ and $r = 3\csc\theta$

To convert equations from Rectangular to Polar:

3. Replace $x$ with $r\cos\theta$
4. Replace $y$ with $r\sin\theta$

Ex. Convert each of the following equations to Polar:

1. $x + y = 5$

$$r\cos\theta + r\sin\theta = 5$$
$$r(\cos\theta + \sin\theta) = 5$$
$$r = \frac{5}{\cos\theta + \sin\theta}$$

2. $(x-1)^2 + y^2 = 1$

$$x^2 - 2x + 1 + y^2 = 1$$
$$x^2 + y^2 - 2x = 0$$
$$r^2 - 2r\cos\theta = 0$$
$$r(r - 2\cos\theta) = 0$$
$$r = 0 \qquad r - 2\cos\theta = 0$$
$$r = 2\cos\theta$$

*$r$ =0 is a point – called the* *pole*

*Since it satisfies $r$= 2cos Θ at $\theta = \frac{\pi}{2}$ we don't need to include $r$ = 0.*

*So the polar equation is $r = 2\cos\theta$*

3. $y = x^2$

$$\frac{y}{x^2} = 1$$
$$\frac{r\sin\theta}{r^2\cos^2\theta} = 1$$
$$\frac{1}{r} = \frac{\cos^2\theta}{\sin\theta}$$
$$r = \frac{\sin\theta}{\cos^2\theta}$$

4. $xy = 1$

$$(r\cos\theta)(r\sin\theta) = 1$$
$$r^2\cos\theta\sin\theta) = 1$$
$$r^2 = \frac{1}{\cos\theta\sin\theta}$$
$$r^2 = \sec\theta\csc\theta$$

To convert equations from Polar to Rectangular:
Use the formulas

$x = r\cos\theta \qquad y = r\sin\theta$

$x^2 + y^2 = r^2 \qquad \tan\theta = \frac{y}{x}$

Ex. Convert each of the following polar equation to rectangular

1. $r = 5$

$$\sqrt{x^2 + y^2} = 5$$
$$x^2 + y^2 = 25$$

2. $\theta = \frac{\pi}{4}$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4}$$
$$\frac{y}{x} = \tan\frac{\pi}{4}$$
$$\frac{y}{x} = 1$$
$$y = x$$

3. $r = 3\csc\theta$

$$r\sin\theta = 3$$
$$y = 3$$

4. $r = -6\cos\theta$

$$\sqrt{x^2 + y^2} = \frac{-6y}{r}$$
$$\sqrt{x^2 + y^2} = \frac{-6y}{\sqrt{x^2+y^2}}$$
$$x^2 + y^2 = -6y$$
$$x^2 + y^2 + 6y = 0$$

**Homework: Day 1: pg. 672 – 673 #1 – 20, 23, 26**
**Day 2: pg. 673 #33 – 72 (3's)**

**If all done in 1 day:**

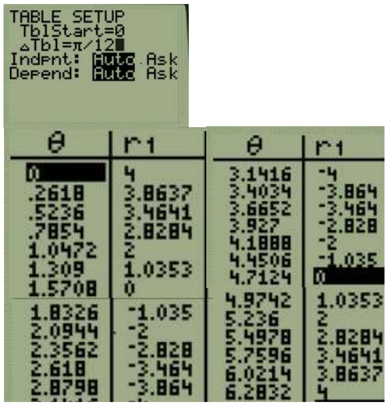
**pg. 672–673 #1-25odd, 33-72(3's)**

**6.4 Graphs of Polar Equations**

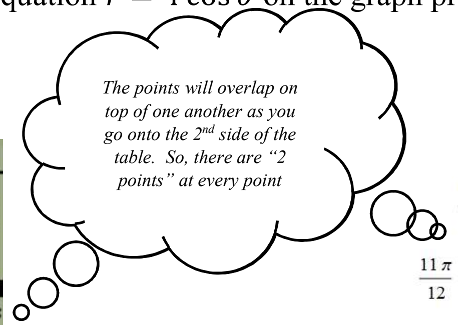
To graph a polar equation:

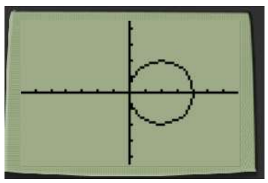
1. Make a table containing the angles and find $r$ based on the value of θ.
2. Plot each point on the graph paper.

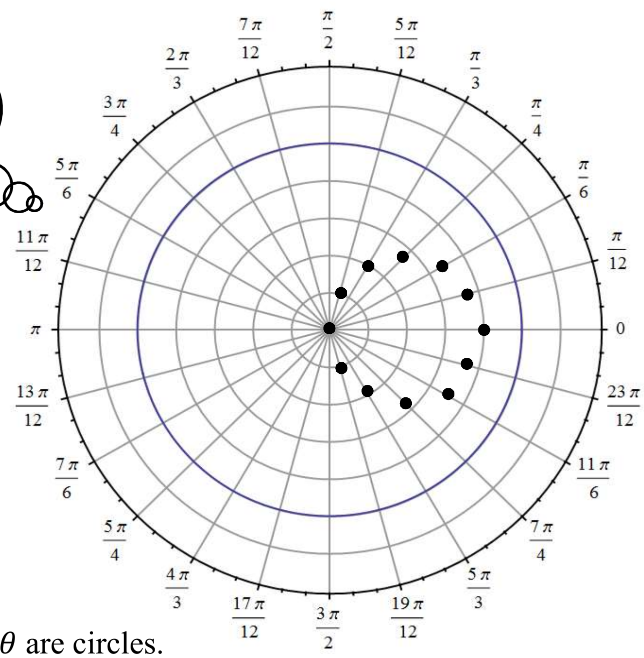
Ex. Graph the polar equation $r = 4\cos\theta$ on the graph provided. Express the equation in rectangular.




| θ | r1 | θ | r1 |
|---|---|---|---|
| 0 | 4 | 3.1416 | -4 |
| .2618 | 3.8637 | 3.4034 | -3.864 |
| .5236 | 3.4641 | 3.6652 | -3.464 |
| .7854 | 2.8284 | 3.927 | -2.828 |
| 1.0472 | 2 | 4.1888 | -2 |
| 1.309 | 1.0353 | 4.4506 | -1.035 |
| 1.5708 | 0 | 4.7124 | 0 |
| 1.8326 | -1.035 | 4.9742 | 1.0353 |
| 2.0944 | -2 | 5.236 | 2 |
| 2.3562 | -2.828 | 5.4978 | 2.8284 |
| 2.618 | -3.464 | 5.7596 | 3.4641 |
| 2.8798 | -3.864 | 6.0214 | 3.8637 |
| | | 6.2832 | 4 |

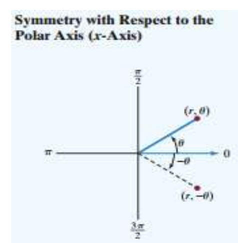




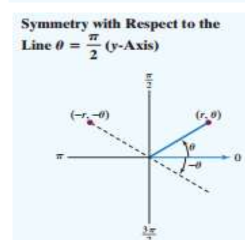

**Property**: The graphs of $r = a\cos\theta$ and $r = a\sin\theta$ are circles.

($a$ is the diameter of the circle)

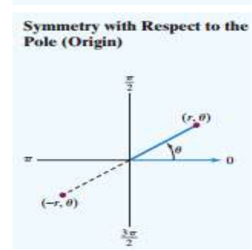



**Symmetry in Polar Coordinates: (ALWAYS CHECK!)**

1. **Symmetry with Respect to the Polar Axis (*x*-axis):**
   *Replace $\theta$ with $-\theta$. If an equivalent equation results, it is symmetric with respect to polar axis*




2. **Symmetry with Respect to the Line $\boldsymbol{\theta = \frac{\pi}{2}}$ (The *y*-axis)**
   *Replace (r,θ) with* $(-r, -\theta)$. If an equivalent equation results, then it is symmetric over $\theta = \frac{\pi}{2}$



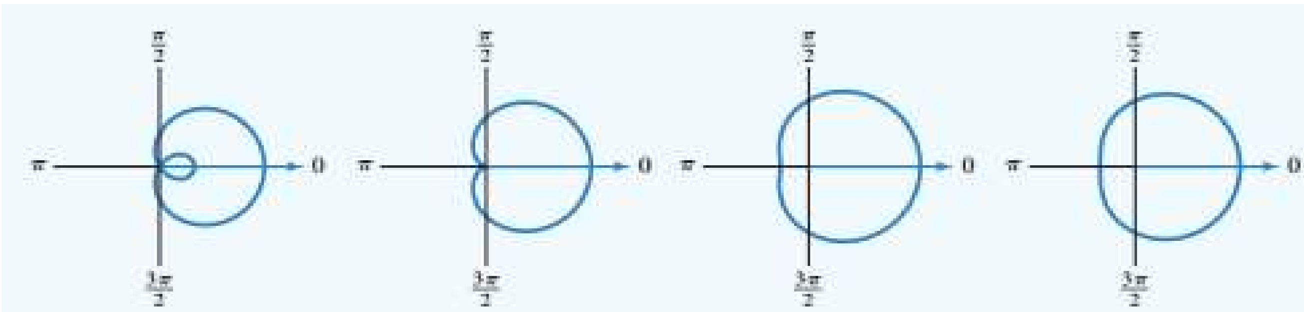

3. **Symmetry with Respect to the Pole (Origin)**
   *Replace $r$ with $-r$. If an equivalent equation results, the graph is symmetric*

**<u>Def</u>:** A **<u>limiçon</u>** is a graph with the following equation:

$$r = a \pm b\sin\theta \quad \text{or} \quad r = a \pm b\cos\theta \quad \text{for } a > 0 \text{ and } b > 0$$

The ratio $\frac{a}{b}$ determines the limiçon's shape:

$\frac{a}{b} < 1$ | $\frac{a}{b} = 1$ | $1 < \frac{a}{b} < 2$ | $\frac{a}{b} \geq 2$



Inner Loop | Heart Shaped (Cardioid) | Dimpled (no inner loop) | No dimple or inner loop

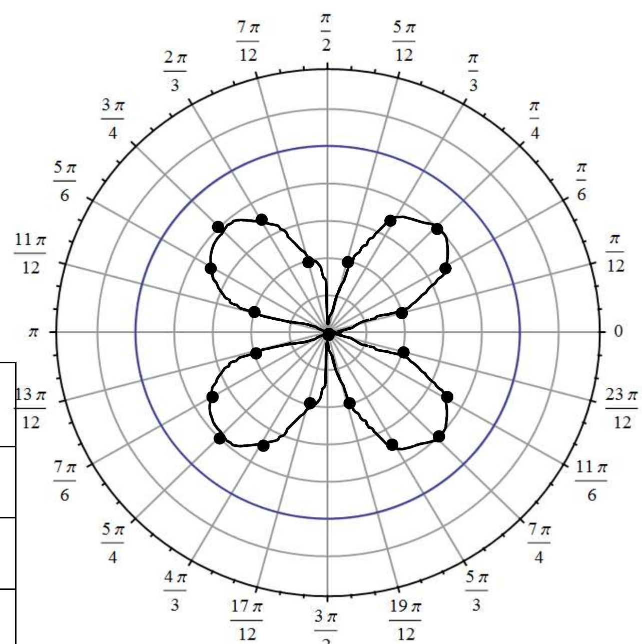
Ex. Graph the polar equation: $r = 4\sin 2\theta$

(i) Check the symmetry

- Test $-\theta$
  - $r = 4\sin(-2\theta) = -4\sin 2\theta$
    *Not same - FAILS*
- Test $-r$ and $-\theta$
  - $-r = 4\sin(-2\theta) \rightarrow -r = -4\sin 2\theta$
    *so* $r = 4\sin 2\theta$ SAME → Symmetry over $y$
- Test $-r$
  - $-r = 4\sin 2\theta \rightarrow r = -4\sin 2\theta$
    Not Same - FAILS

(ii) Make a table for $0 \leq \theta \leq \frac{\pi}{2}$

| $\theta$ | 0 | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{5\pi}{12}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $r$ | 0 | 2 | 3.464 | 4 | 3.464 | 2 | 0 |
| $\theta$ | $\frac{7\pi}{12}$ | $\frac{2\pi}{36}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\frac{11\pi}{12}$ | $\pi$ | |
| $r$ | –2 | –3.464 | –4 | –3.464 | –2 | 0 | |




(iii) Plot these points and use the symmetry found to plot the remaining points.

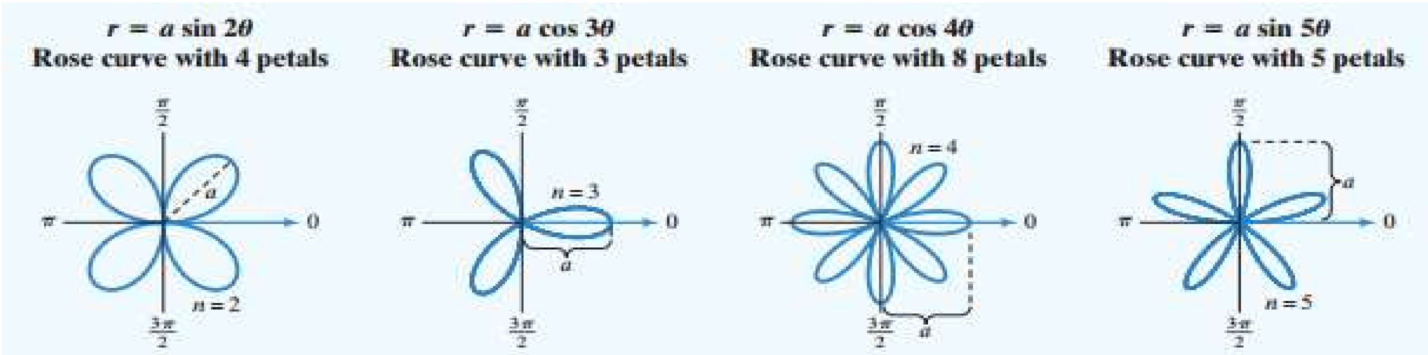
*The resulting table graphs the "petals" on top right and bottom right quadrants above.*

*By symmetry over the "ý-axis" we get the other 2 petals.*

**<u>Def</u>**: A **<u>rose curve</u>** is a polar graph of the form:

$$r = a\sin n\theta \quad \text{or} \quad r = a\cos n\theta \quad \text{for } a \neq 0$$

If *n* is *even*, the rose will have 2 petals. If *n* is *odd*, then it will have *n* petals.




Ex. Graph the polar equation: $r^2 = 4\sin 2\theta$

(i) Check the symmetry

- Test $-\theta$
- Test $-r$ and $-\theta$
- Test $-r$

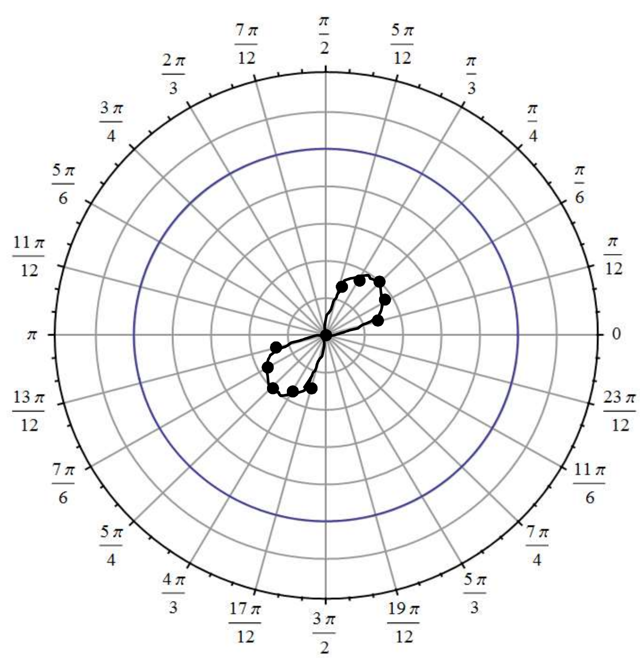
- $r^2 = 4\sin(-2\theta) = -4\sin 2\theta$
  *Not the same ⁻ no symmetry over x*
- $(-r)^2 = 4\sin(-2\theta) \rightarrow r^2 = -4\sin 2\theta$
  *Not the same ⁻ no symmetry over y*
- $(-r)^2 = 4\sin(2\theta) \rightarrow r^2 = 4\sin 2\theta$
  same– so Pole (Origin) Symmetry

(ii) Make a table for $0 \leq \theta \leq \frac{\pi}{2}$

| $\theta$ | 0 | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{5\pi}{12}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $r$ | 0 | 1.414 | 1.861 | 2 | 1.861 | 1.414 | 0 |

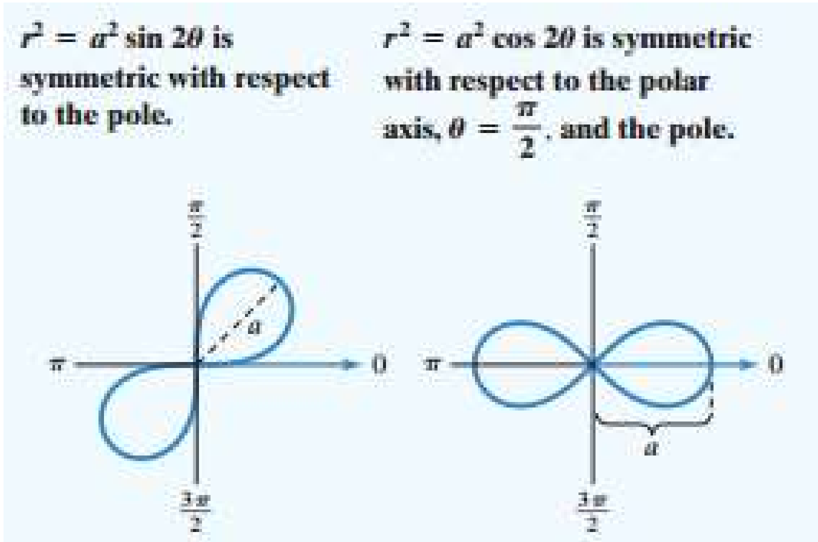
*If you attempt to create the table in quadrant II, all the $4\sin 2\theta$ values are negative. This can never be equal to $r^2$, therefore they would not exist.*

*Using the symmetry given in part (i), we get the graph to the right.*



**<u>Def</u>:** A **<u>lemniscate</u>** is a polar graph of the form:

$$r^2 = a^2 \sin 2\theta \qquad \text{or} \qquad r^2 = a^2 \cos 2\theta \quad \text{with } a \neq 0$$




Propellars

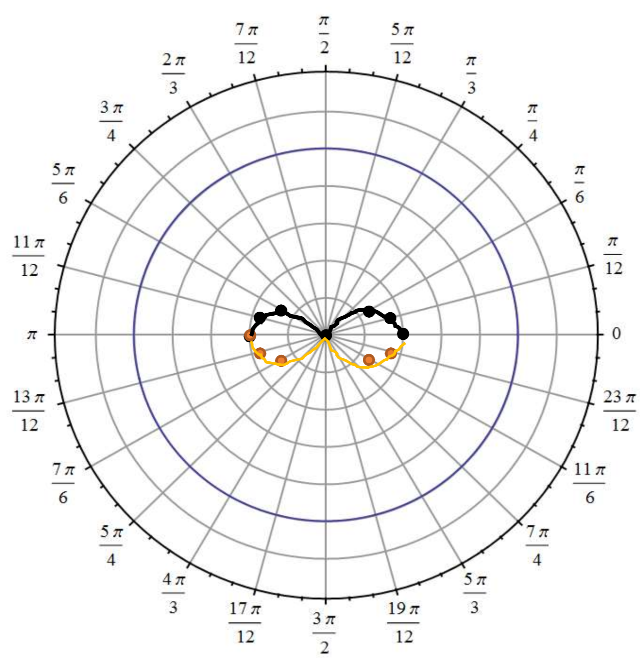
Ex. Graph the polar equation $r^2 = 4\cos 2\theta$

(i) Check the symmetry

- Test $-\theta$
    - $r^2 = 4\sin(-2\theta) = -4\sin 2\theta$
      *Not the same – no symmetry over x*
- Test $-r$ and $-\theta$
    - $(-r)^2 = 4\sin(-2\theta) \rightarrow r^2 = -4\sin 2\theta$
      *Not the same – no symmetry over y*
- Test $-r$
    - $(-r)^2 = 4\sin(2\theta) \rightarrow r^2 = 4\sin 2\theta$
      same – so Pole (Origin) Symmetry

(ii) Make a table for $0 \leq \theta \leq \frac{\pi}{2}$

| $\theta$ | $0$ | $\frac{\pi}{12}$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{5\pi}{12}$ | $\frac{\pi}{2}$ |
|---|---|---|---|---|---|---|---|
| $r$ | 2 | 1.861 | 1.414 | 0 | Error | Error | Error |
| $\theta$ | $\frac{7\pi}{12}$ | $\frac{2\pi}{36}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\frac{11\pi}{12}$ | $\pi$ | |
| $r$ | Error | Error | 0 | 1.414 | 1.861 | 2 | |




*The table above graphs the black curve shown to the right. The "Error" values in the table appear because* $4\cos 2\theta$ *is negative for these values.* $r^2$ *cannot be negative, therefore the "Error". Once the curve from 0 to π is created, then using symmetry (only point – over the origin), we get the orange points and curve.*

**Homework: pg. 683 #1-6, 7, 10, 12, 15-30 (5's)**

## 6.5 Complex Numbers in Polar Form: DeMoivre's Theorem

Recall: The number, ***i***, is an imaginary number defined $i = \sqrt{-1}$

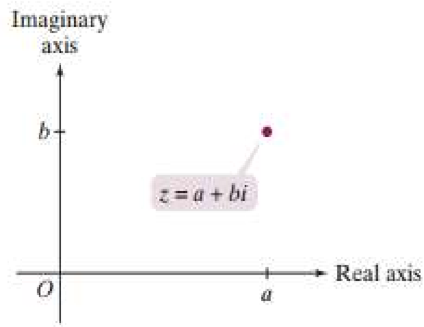
*Note: Numbers that contain i are called* ***complex numbers***

**Def:** A **complex number** is a numerical expression of the form: $a + bi$ for any real numbers $a$ and $b$, with $b \neq 0$

Ex. $6 + 3i \qquad -4 - \frac{3}{4}i \qquad i$

**Def:** The **complex plane** is a coordinate system consisting of two axes.

- The horizontal axis is called the **real axis**.
- The vertical axis is called the **imaginary axis**.
- Every complex number corresponds to a point in the complex plane
- To plot a complex number in the complex plane:
  1. Go along the real axis and find $a$.
  2. Go up the imaginary axis to the coefficient of $i$ and place the point.

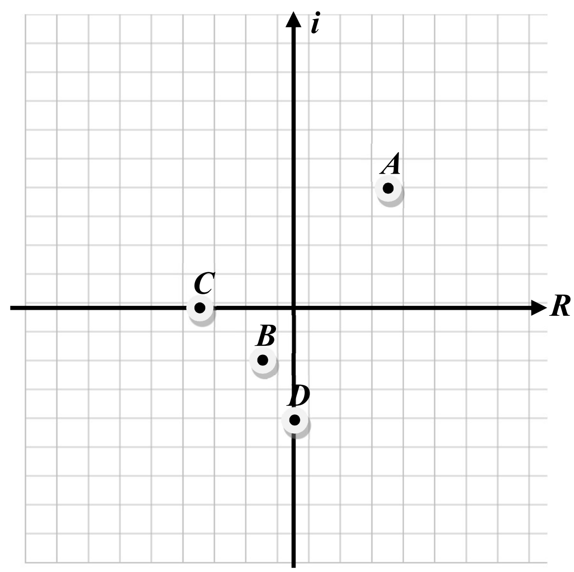



Ex. On the complex plane to the right, plot and label each of the following complex numbers

A: $z = 3 + 4i$
B : $z = -1 - 2i$
C: $z = -3$
D: $z = -4i$

*Graph the following points as well:*

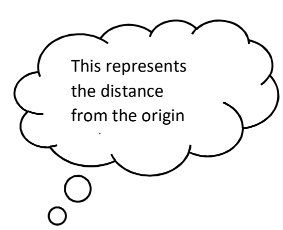
*E: z = 2 + 3i*
*F: z = –1 – 2i*
*G: z = –3*
*H: z = –4i*




**Th: The Absolute Value of a Complex Number**

The **absolute value** of a complex number $a + bi$ is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}$$




Ex. Determine the absolute value of each of the following complex numbers

(a) $z = 3 + 4i$

$|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16}$

$|z| = \sqrt{25}$

$|z| = 5$

(b) $z = -1 - 2i$

$|z| = \sqrt{(-1)^2 + (-2)^2} = \sqrt{1 + 4}$

$|z| = \sqrt{5}$

**Th: Polar Form of a Complex Number**

The complex number $z = a + bi$ is written in *polar form* as:

*Use absolute values to find the reference angle, then use a and b to determine auad and*

$$z = r(\cos\theta + i\sin\theta)$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$r$ is called the **modulus** of the complex number $z$ and the angle $\theta$ is called the **argument** of the complex number with $0 \le \theta \le 2\pi$

ex. Write the following complex numbers in polar form:

(a) $z = -2 - 2i$

$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{4+4} = \sqrt{8}$

$\theta = \tan^{-1}\left(\frac{-2}{-2}\right) = 45 \leftarrow ref\ angle$

Quad III ($a$ and $b$ both negative)

$\therefore\ \theta = 225 \ \rightarrow\ \theta = \frac{5\pi}{4}$

$z = \sqrt{8}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

(b) $z = -1 + i\sqrt{3}$

$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{9} = 3$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60 \leftarrow ref\ angle$

Quad II ($a$ - and $b$+)

$\therefore\ \theta = 120 \ \rightarrow\ \theta = \frac{2\pi}{3}$

$z = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

(c) $z = 3i$

$r = \sqrt{(0)^2 + (3)^2} = \sqrt{9} = 3$

$\theta = \tan^{-1}\left(\frac{3}{0}\right) = \tan^{-1}(\infty) = 90$

$\therefore\ \theta = \frac{\pi}{2}$

$z = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Ex. Write $z = 2(\cos 60 + i\sin 60)$ in rectangular form

*Simply calculate the values:*

$$z = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

*Distributing.....*

$z = 1 + i\sqrt{3}$

Ex. Write $z = 4(\cos 30 + i\sin 30)$ in rectangular form

*Simply calculate the values:*

$$z = 4\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

*Distributing.....*

$z = 2\sqrt{3} + 2i$

**Th: The Product of Two Complex Number in Polar Form**

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their product $z_1z_2$ is

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

*To multiply two complex numbers, multiply the moduli(radii) and add the arguments(angles)*

Ex. Find the product of $z_1 = 4(\cos 50 + i\sin 50)$ and $z_2 = 7(\cos 100 + i\sin 100)$

$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] = 4(7)[\cos(50 + 100) + i\sin(50 + 100)]$

$z_1z_2 = 28(\cos 150 + i\sin 150) \ \rightarrow\ z_1z_2 = 28\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) \ \rightarrow\ z_1z_2 = -14\sqrt{3} + 14i$

Ex. Find the product of $z_1 = 6(\cos 40 + i\sin 40)$ and $z_2 = -5(\cos 20 + i\sin 20)$

$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] = 6(-5)[\cos(40 + 20) + i\sin(40 + 20)]$

$z_1z_2 = -30(\cos 60 + i\sin 60) \ \rightarrow\ z_1z_2 = -30\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \ \rightarrow\ z_1z_2 = -15 - 15i\sqrt{3}$

**Th: The Quotient of Two Complex Numbers in Polar Form**

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their quotient, $\frac{z_1}{z_2}$ is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

*To divide two complex numbers, divide the moduli(radii) and subtract the arguments(angles)*

Ex. Find the quotient $\frac{z_1}{z_2}$ if the complex numbers. Leave the answer in polar form

$$z_1 = 12(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) \qquad z_2 = 4(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] = \frac{12}{3}\left[\cos\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) + i\sin\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)\right]$

$\frac{z_1}{z_2} = 4\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$ $\rightarrow$ $\frac{z_1}{z_2} = 4[0 + i(1)]$ $\rightarrow$ $\frac{z_1}{z_2} = 0 + 4i$

Ex. Find the quotient $\frac{z_1}{z_2}$ if the complex numbers. Leave the answer in polar form

$$z_1 = 50(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) \qquad z_2 = 5(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] = \frac{50}{5}\left[\cos\left(\frac{4\pi}{3} - \frac{\pi}{3}\right) + i\sin\left(\frac{4\pi}{3} - \frac{\pi}{3}\right)\right]$

$\frac{z_1}{z_2} = 10\left[\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right] = 10(\cos\pi + i\sin\pi)$ $\rightarrow$ $\frac{z_1}{z_2} = 10[-1 + i(0)]$ $\rightarrow$ $\frac{z_1}{z_2} = -10 + 0i$

**Powers of Complex Numbers in Polar Form – DeMoivre's Theorem**

Let $z = r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If *n* is a positive integer, then *z* to the *n*th power, $z^n$, is

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Ex. Find $[2(\cos 20 + i\sin 20)]^6$. Write the answer in rectangular form, *a* + *bi*

$$z^n = r^n(\cos n\theta + i\sin n\theta) = 2^6[\cos 6(20) + i\sin 6(20)] = 64(\cos 120 + i\sin 120)$$
$$[2(\cos 20 + i\sin 20)]^6 = 64\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -32 + 32i\sqrt{3}$$

Ex. Find $[2(\cos 30 + i\sin 30)]^5$. Write the answer in rectangular form, *a* + *bi*

$$z^n = r^n(\cos n\theta + i\sin n\theta) = 2^5[\cos 5(30) + i\sin 5(30)] = 32(\cos 150 + i\sin 150)$$
$$[2(\cos 30 + i\sin 30)]^5 = 32\left(-\frac{\sqrt{3}}{2} + i\left(\frac{1}{2}\right)\right) = -16\sqrt{3} + 16i$$

Ex. Find $(1 + i)^6$ using DeMoivre's Theorem. Write the answer in rectangular form, *a* + *bi*.

*Convert to Polar Form:*

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

*Ref* $\angle\theta = \tan^{-1}\left(\frac{1}{1}\right) = 45$

*Quad I (a and b both +)*

$\theta = \frac{\pi}{4}$

$(1 + i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$(1 + i)^6 = (\sqrt{2})^6\left[\cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4}\right] = 8\big(0 + i(-1)\big) = 0 - 8i$ or $-8i$

Ex. Find $(1+i)^4$ using DeMoivre's Theorem. Write the answer in rectangular form, $a + bi$.

*Convert to Polar Form:*
$r = \sqrt{1^2 + 1^2} = \sqrt{2}$
*Ref* $\angle \theta = \tan^{-1}\left(\frac{1}{1}\right) = 45$
*Quad I (a and b both +)*
$\theta = \frac{\pi}{4}$

$(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
$(1+i)^4 = (\sqrt{2})^4\left[\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right] = 4\left(-1 + i(0)\right) = -4 + 0i$ or $-4$

Since $[2(\cos 20 + i\sin 20)]^6 = 64(\cos 120 + i\sin 120)$, then $2(\cos 20 + i\sin 20)$ is called the **complex sixth root** of $64(\cos 120 + i\sin 120)$.

**Th: DeMoivre's Theorem for Finding Complex Roots**

Let w= $r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If $w \neq 0$, $w$ has $n$ distinct complex $n$th roots given by:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \text{ radians}$$

or

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360° k}{n}\right) + i\sin\left(\frac{\theta + 360° k}{n}\right)\right] \text{ degrees}$$

Where $k = 0, 1, 2, \ldots, n-1$

Ex. Find all the complex fourth roots of $16(\cos 120 + i\sin 120)$. Write the roots in polar form with $\theta$ in degrees.

$z_4 = \sqrt[4]{16}\left[\cos\left(\frac{120 + 360k}{4}\right) + i\sin\left(\frac{120 + 360k}{4}\right)\right]$ , $k = 0, 1, 2, 3$
$z_4 = 2[\cos(30 + 90k) + i\sin(30 + 90k)]$ , $k = 0, 1, 2, 3$

$k = 0$: $z_0 = 2[\cos(30 + 0) + i\sin(30 + 0)] = 2(\cos 30 + i\sin 30)$ → $2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \sqrt{3} + i$

$k = 1$: $z_1 = 2[\cos(30 + 90) + i\sin(30 + 90)] = 2(\cos 120 + i\sin 120)$ → $2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$

$k = 2$: $z_2 = 2[\cos(30 + 180) + i\sin(30 + 180)] = 2(\cos 210 + i\sin 210)$ → $2\left(-\frac{\sqrt{3}}{2} + i\left(-\frac{1}{2}\right)\right) = -2\sqrt{3} - i$

$k = 3$: $z_3 = 2[\cos(30 + 270) + i\sin(30 + 270)] = 2(\cos 300 + i\sin 300)$ → $2\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) = 1 - i\sqrt{3}$

Ex. Find all the cube roots of 8. Write the roots in rectangular form.

$8 = 8 + 0i$
$r = \sqrt{8^2 + 0^2} = 8$

$\theta = \tan^{-1}\left(\frac{0}{8}\right) = 0$

$z = 8(\cos 0 + i\sin 0)$

$z_3 = \sqrt[3]{8}\left(\cos\frac{0 + 360k}{3} + i\sin\frac{0 + 360k}{3}\right) = 2(\cos 120k + i\sin 120k)$ , *for* $k = 0, 1, 2$

$k = 0$: $z_0 = 2(\cos 0 + i\sin 0)$ → $z_0 = 2(1 + 0i)$ → $z_0 = 2$

$k = 1$: $z_1 = 2(\cos 120 + i\sin 120)$ → $z_1 = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$ → $z_1 = -1 + i\sqrt{3}$

$k = 2$: $z_2 = 2(\cos 240 + i\sin 240)$ → $z_1 = 2\left(-\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$ → $z_2 = -1 - i\sqrt{3}$

***Have students do all the 4th roots of 16 for more practice***

**Homework: pg. 696-698 #3-36(3's), 37, 43, 44, 46, 47, 53, 57, 65, 68, 72,**

## 6.6 Vectors

**<u>Def</u>:** Quantities that involve both a magnitude and a direction are called ***vector quantities*** or **<u>vectors</u>**.

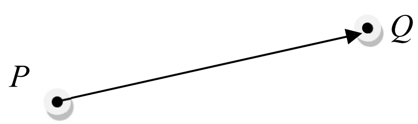
Ex. Velocity, force, acceleration

*Vectors are normally denoted with boldface with an arrow over a single letter. Ex.* $\vec{\mathbf{v}}$

**<u>Def</u>**: Quantities that only involve a magnitude are called ***scalar quantities*** or **<u>scalars</u>**.

Ex. Temperature, lengths, area, …

**<u>Def</u>**: A line segment to which a direction has been assigned is called a **<u>directed line segment</u>**.

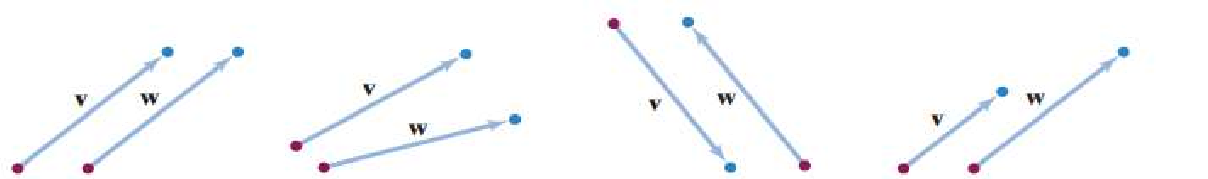



This segment is denoted $\overrightarrow{PQ}$ (from $P$ to $Q$)

$P$ is called the **<u>initial point</u>**(or ***tail***) and $Q$ is called the **<u>terminal point</u>**(or ***head***).

**<u>Def</u>**: The **<u>magnitude</u>** of a directed segment, $\overrightarrow{PQ}$, is its length. This is denoted, $\|\overrightarrow{PQ}\|$. It is the distance from $P$ to $Q$. Magnitudes are **NEVER** negative.

Two vectors, $\vec{v}$ and $\vec{w}$ have 4 possible relationships:




**(a)** $\mathbf{v} = \mathbf{w}$ because the vectors have the same magnitude and same direction.

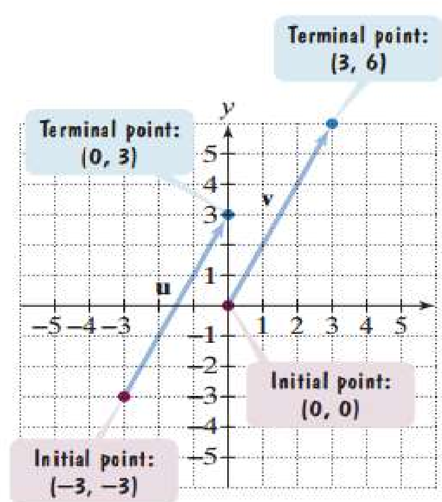
**(b)** Vectors **v** and **w** have the same magnitude, but different directions.

**(c)** Vectors **v** and **w** have the same magnitude, but opposite directions.

**(d)** Vectors **v** and **w** have the same direction, but different magnitudes.

Ex. Use the figure to the right to show **u = v**

**(to show same direction – show same slope)**




*Direction:*

$m_u = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-(-3)}{0-(-3)} = \dfrac{6}{3} = 2 \qquad m_v = \dfrac{6-0}{3-0} = \dfrac{6}{3} = 2$

*Same slope* → *Same direction*

*Magnitude*

$||\boldsymbol{u}|| = \sqrt{(0-(-3))^2 + (3-(-3))^2} = \sqrt{9+36} = \sqrt{45}$

$||\boldsymbol{v}|| = \sqrt{(3-0)^2 + (6-0)^2} = \sqrt{9+36} = \sqrt{45}$

*Equal* → *Same magnitude*

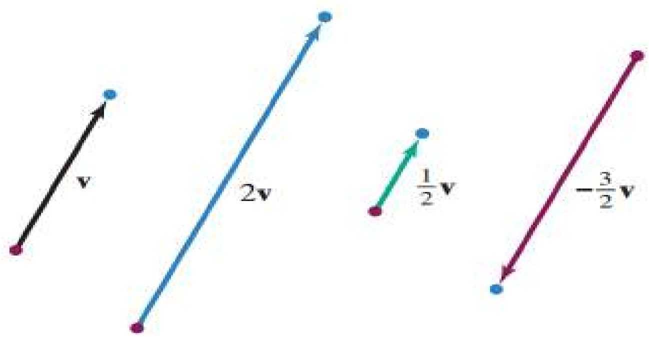
*Since both same direction and magnitude:* ***u = v***

**Property:** Scalar Multiplication

**If *k* is a real number and v is vector, then *k*v is called <u>scalar multiple</u> of the vector v.** The magnitude and direction of $k\mathbf{v}$ is determined as follows:

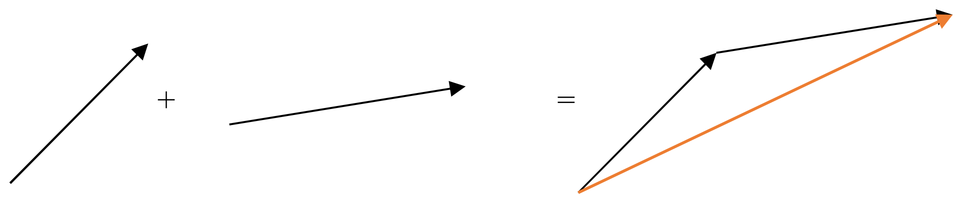
- The magnitude of $k\mathbf{v}$ is $|k| \times \|\mathbf{v}\|$ (absolute value of $k$ times magnitude of $\mathbf{v}$)
- If $k > 0$, then the direction remains the same
- If $k < 0$, then the direction is opposite the original direction

Ex.

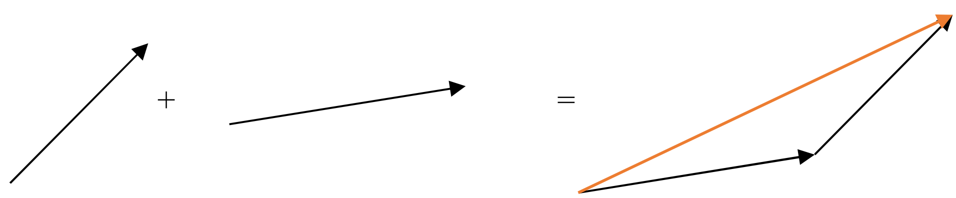



**<u>Adding Vectors</u>:**

Geometrically, you ***add vectors*** by connecting the head of one vector to the tail of another:



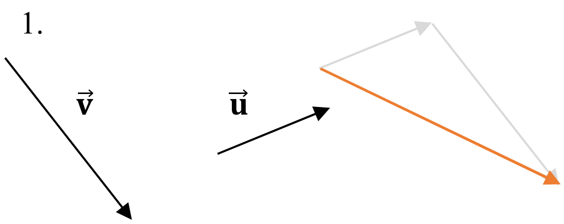

The dashed line is the sum of the two vector, called the ***<u>resultant vector</u>***. It does not matter which order you connect the vectors:



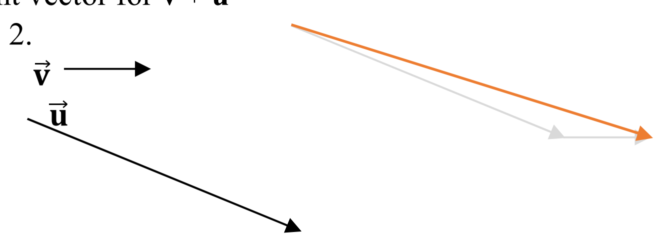

Your resultant vector will still have the same magnitude and direction.

Ex. Given vectors $\mathbf{v}$ and $\mathbf{u}$, draw the resultant vector for $\mathbf{v} + \mathbf{u}$

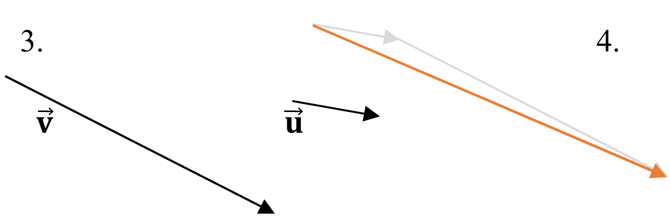
1.



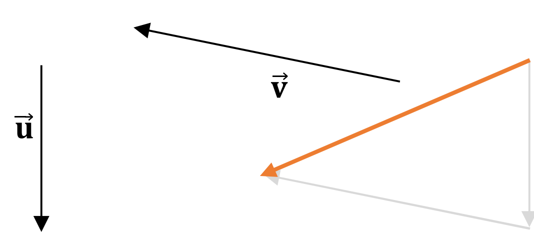

2.




3.




4.

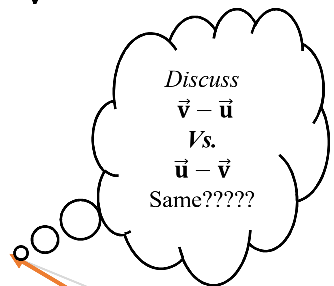


**Difference of Two Vectors, $\vec{\mathbf{v}} - \vec{\mathbf{u}}$**

$\vec{\mathbf{v}} - \vec{\mathbf{u}} = \vec{\mathbf{v}} + (-\vec{\mathbf{u}})$ – The difference of two vectors is the same as adding the scalar opposite of **u**.

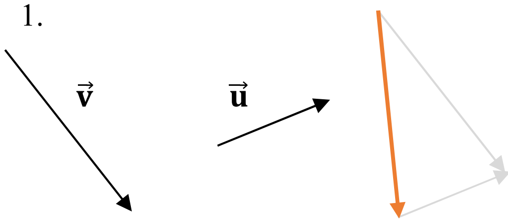
To do this geometrically (quickly): Put the head of **u** at the head of **v** and connect the tail of **v** to the tail of **u**:


By putting the head of **u** at the head of **v**, you create –**u**

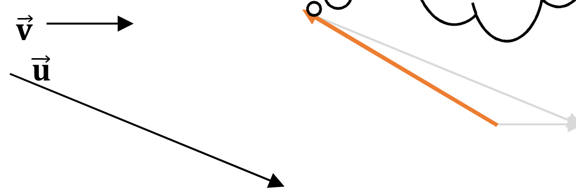



Ex. Given vectors **v** and **u**, draw the resultant vector for $\vec{\mathbf{v}} - \vec{\mathbf{u}}$

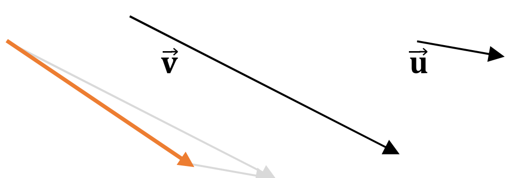
1.



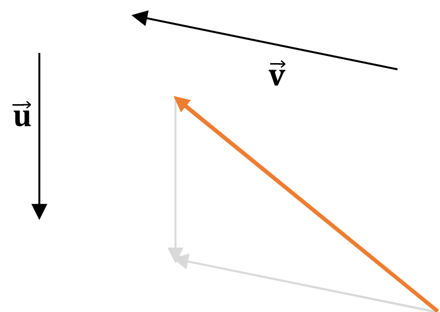

2.




3.




4.

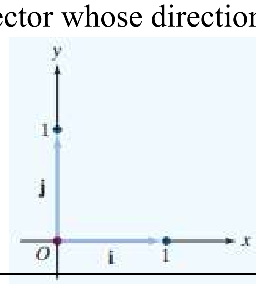



**Vectors in the Rectangular Coordinate System**

- Vectors can be shown in the rectangular coordinate system

**<u>Def</u>: The i and j Unit Vectors**

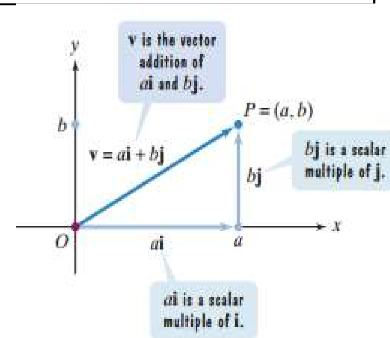
Vector **i** is the unit vector whose direction is along the positive $x$-axis with a magnitude of 1.
Vector **j** is the unit vector whose direction is along the positive $y$-axis with a magnitude of 1.




All vectors can be represented in terms of **i** and **j**.

Consider vector **v** with initial point at (0,0) and terminal point at $P(a, b)$.

We can present **v** using **i** and **j as** $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$



**<u>Property</u>: Representing Vectors in Rectangular Coordinates**

Vector **v**, from (0,0) to $(a, b)$, is represented by

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j}$$

The real numbers $a$ and $b$ are called the **<u>scalar components of v</u>.**

- $a$ is the **<u>horizontal compenent</u>** of **v**
- $b$ is the **<u>vertical component</u>** of **v**

The vector sum $a\mathbf{i} + b\mathbf{j}$ is called the **<u>linear combination</u>** of vectors **i** & **j**

The magnitude of $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$

Ex. (a) One the set of axes, sketch each vector and find its magnitude.

$\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$ **(red)**

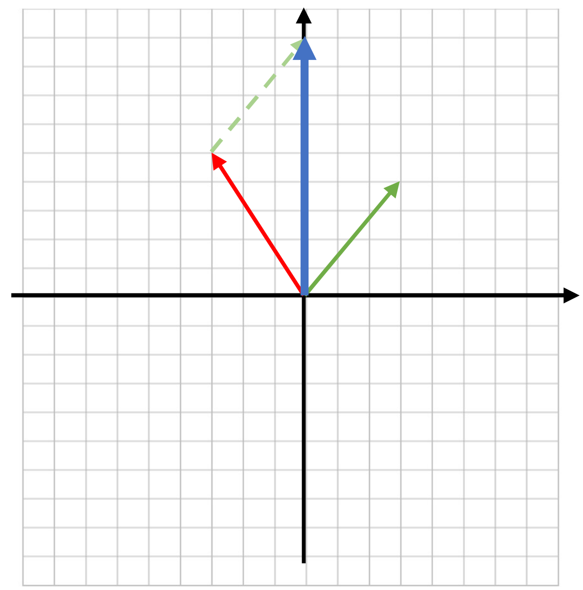
$\mathbf{u} = 3\mathbf{i} + 3\mathbf{j}$ **(green)**

(b) On the axes, sketch **w** = **v** + **u** **(blue)**

(c) What is linear combination and magnitude of **w**?

$\boldsymbol{w} = 0\mathbf{i} + 7\mathbf{j}$

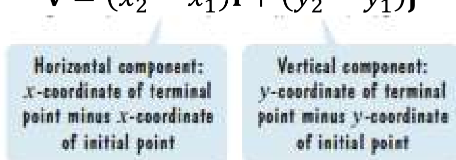
$\|\boldsymbol{w}\| = \sqrt{0^2 + 7^2} = 7$



**<u>Property</u>: Representing Vectors in Rectangular Coordinates**

Vector **v** with initial point $P_1(x_1,y_1)$ and terminal point $P_2(x_2,y_2)$ is equal to the vector

$$\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$$




Ex. Write vector **v** in terms of **i** and **j** if initial point $P_1=(3, -1)$ and terminal point $P_2= (-2,5)$. What is the magnitude?

$\mathbf{v} = (-2-3)\mathbf{i} + \big(5-(-1)\big)\mathbf{j} \quad \rightarrow \quad \mathbf{v} = -5\mathbf{i} + 6\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{(-5)^2 + 6^2} = \sqrt{25+36} = \sqrt{41}$

**Operations with Vectors in Terms of i and j**

1. **Adding/Subtracting Vectors**

   If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$, then

   $$\mathbf{v} \pm \mathbf{w} = (a_1 \pm a_2)\mathbf{i} + (b_1 \pm b_2)\mathbf{j}$$

   ex. If $\mathbf{v} = 7\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - 5\mathbf{j}$, find $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$.

   $$\mathbf{v} + \mathbf{w} = (7+4)\mathbf{i} + (3-5)\mathbf{j} \quad \rightarrow \quad \mathbf{v} + \mathbf{w} = 3\mathbf{i} - 2\mathbf{j}$$
   $$\mathbf{v} - \mathbf{w} = (7-4)\mathbf{i} + \big(3-(-5)\big)\mathbf{j} \quad \rightarrow \quad \mathbf{v} - \boldsymbol{w} = 3\mathbf{i} + 8\mathbf{j}$$

2. **Scalar Multiplication with Vectors**

   If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ and $k$ is a real number, then the scalar multiplication of the vector **v** and that scalar $k$ is

   $$k\mathbf{v} = ka\mathbf{i} + kb\mathbf{j}$$

   ex. If $\mathbf{v} = 5\mathbf{i} + 4\mathbf{j}$, find

   (a) $6\mathbf{v}$ (b) $-3\mathbf{v}$

   $$6\mathbf{v} = 30\mathbf{i} + 26\mathbf{j} \qquad\qquad -3\mathbf{v} = -15\mathbf{i} - 12\mathbf{j}$$

   ex. If $\mathbf{v} = 7\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - 5\mathbf{j}$, find $6\mathbf{v} + 5\mathbf{w}$ and $3\mathbf{v} - 2\mathbf{w}$.

   $$6\mathbf{v} + 5\mathbf{w} = (42\mathbf{i} + 18\mathbf{j}) + (20\mathbf{i} - 25\mathbf{j}) = 62\mathbf{i} - 7\mathbf{j}$$

   $$3\mathbf{v} - 2\mathbf{w} = (21\mathbf{i} + 9\mathbf{j}) - (8\mathbf{i} - 10\mathbf{j}) = 13\mathbf{i} + 19\mathbf{j}$$

**Def: The Zero Vector**

A vector whose magnitude is 0 is called the **zero vector, 0**. The zero vector is assigned no direction. In terms of **i** and **j**: $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$

**Properties of Vector Addition and Scalar Multiplication**

If **u**, **v**, and **w** are vectors and $c$ and $d$ are scalars, then the following properties are true:

1. **Commutative Property:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. **Associative Property of Addition:** $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. **Additive Identity:** $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$
4. **Additive Inverse:** $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
5. **Associative Property of Multiplication:** $(cd)\mathbf{u} = c(d\mathbf{u})$
6. **Distributive Property:**
   - $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
   - $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. **Multiplicative Identity:** $1\mathbf{u} = \mathbf{u}$
8. **Multiplication Property of Zero:** $0\mathbf{u} = 0$
9. **Magnitude Property:** $\|c\mathbf{v}\| = |c|\cdot\|\mathbf{v}\|$

## Unit Vectors

Def: A **unit vector** is a vector whose magnitude is one.

- In many applications of vectors, it is helpful to find the unit vector that has the same direction as a given vector.

Property: For any nonzero vector **v**, the unit vector is equal to: $\frac{\mathbf{v}}{\|\mathbf{v}\|}$

Ex. Find the unit vector in the same direction $\mathbf{v} = \mathbf{5}\mathrm{i} - 12\mathbf{j}$

*Find the magnitude:* $\|v\| = \sqrt{a^2 + b^2} = \sqrt{5^2 + (-12)^2} = \sqrt{169} = 13$

*Unit vector is:* $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{5\mathbf{i}-1\ \mathbf{j}}{13} = \frac{5}{13}\mathbf{i} - \frac{12}{13}\mathbf{j}$

**You can verify this by finding the magnitude of the result:**

$$\sqrt{\frac{5}{13}^{2} + \frac{12^{2}}{13}} = \sqrt{\frac{25}{169} + \frac{144}{169}} = \sqrt{\frac{169}{169}} = 1$$

Ex. Find the unit vector in the same direction as $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$. Verify the magnitude is 1.

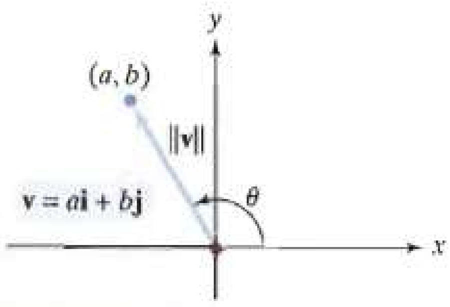
*Find the magnitude:* $\|v\| = \sqrt{a^2 + b^2} = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$

*Unit vector is:* $\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{4\mathbf{i}-3\mathbf{j}}{5} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$

**You can verify this by finding the magnitude of the result:**

$$\sqrt{\frac{4^2}{5} + \left(\frac{-3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}} = 1$$

## Writing a Vector in Terms of Its Magnitude and Direction




Consider the vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$.
*The components a and b can be expressed in terms of the magnitude of vector* $\mathbf{v}$ *and the angle* $\theta$ *that makes* $\theta$ *that* $\mathbf{v}$ *makes with the positive x-axis.*

This angles is called the **direction angle of v** and is shown in the diagram to the left.

By definition of the sine and cosine, we get:

$$\cos\theta = \frac{a}{\|\mathbf{v}\|} \text{ and } \sin\theta = \frac{b}{\|\mathbf{v}\|}$$

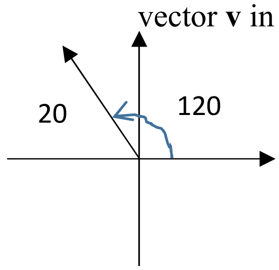
$$a = \|\mathbf{v}\|\cos\theta \text{ and } b = \|\mathbf{v}\|\cos\theta$$

Therefore: $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j}$

Property: Let $\mathbf{v}$ be a nonero vector. If θ is the directional angle measured from the positive *x*-axis to $\mathbf{v}$, then the vector can be expressed in terms of its magnitude and direction angle is

$$\mathbf{v} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j} \quad \textbf{or } \mathbf{v} = \|\mathbf{v}\|(\cos\theta\mathbf{i} + \sin\theta\,\mathbf{j})$$

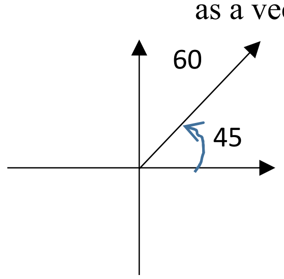
Ex. The wind is blowing at 20 miles per hour in the direction N30°W. Express its velocity as a vector $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.




$$\mathbf{v} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j} = 20\cos 120\,\mathbf{i} + 20\sin 120\,\mathbf{j}$$

$$\mathbf{v} = 20\left(-\tfrac{1}{2}\right)\mathbf{i} + 20\left(\tfrac{\sqrt{3}}{2}\right)\mathbf{j} \quad \rightarrow \quad \mathbf{v} = -10\mathbf{i} + 10\sqrt{3}\,\mathbf{j}$$

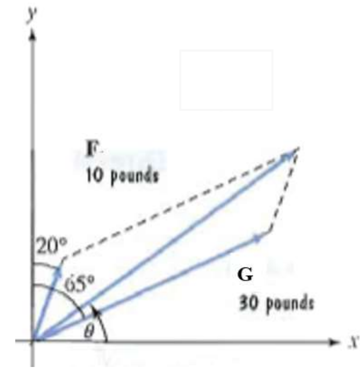
Ex. The jet stream is blowing at 60 miles per hour in the direction N45°E. Express its velocity as a vector $\mathbf{v}$ in terms of $\mathbf{i}$ and $\mathbf{j}$.




$$\mathbf{v} = \|\mathbf{v}\|\cos\theta\,\mathbf{i} + \|\mathbf{v}\|\sin\theta\,\mathbf{j} = 60\cos 45\,\mathbf{i} + 60\sin 45\,\mathbf{j}$$

$$\mathbf{v} = 60\left(\tfrac{\sqrt{2}}{2}\right)\mathbf{i} + 60\left(\tfrac{\sqrt{2}}{2}\right)\mathbf{j} \quad \rightarrow \quad \mathbf{v} = 30\sqrt{2}\,\mathbf{i} + 30\sqrt{2}\,\mathbf{j}$$

Ex. Two forces **F** and **G** of magnitude 10 and 30 pounds respectively act on an object. The direction of **F** is N20°E and the direction of **G** is N65°E. Find the magnitude and direction of the resultant force. Express the magnitude to the nearest hundredth of a pound and direction angle to the nearest tenth of a degree.




**Using directionality: N20°E means 20°E of due North From the positive *x*-axis, it would be 90 – 20 = 70**

**N65°E means 65°E of due North. From the positive *x*-axis, it would be 90 – 65 = 25**

$$\mathbf{F} = \|\mathbf{F}\| \cos\theta\, \mathbf{i} + \|\mathbf{F}\| \sin\theta \mathbf{j} = 10\cos 70\mathbf{i} + 10\sin 70\mathbf{j}$$

$$\mathbf{F} = 10\cos 70\mathbf{i} + 10\sin 70\mathbf{j} \approx 3.42\mathbf{i} + 9.40\mathbf{j}$$

$$\mathbf{G} = \|\mathbf{G}\| \cos\theta\, \mathbf{i} + \|\mathbf{G}\| \sin\theta \mathbf{j} = 30\cos 25\mathbf{i} + 30\sin \mathbf{25j}$$

$$\mathbf{G} = 30\cos 25\mathbf{i} + 30\sin 25\mathbf{j} \approx 27.19\mathbf{i} + 12.68\mathbf{j}$$

**The resultant force, V=F + G is**
**V** = (3.42 + 27.19)**i** + (9.40 + 12.68)**j** = 30.61**i** + 22.08**j**

$$\|\mathbf{V}\| = \sqrt{a^2 + b^2} = \sqrt{(30.61)^2 + (22.08)^2} \approx 37.74 \text{ pounds}$$

$$\theta = \cos^{-1}\frac{a}{\|\mathbf{V}\|} = \cos^{-1}\frac{30.61}{37.74} \approx 35.8°$$

**Study Tip**

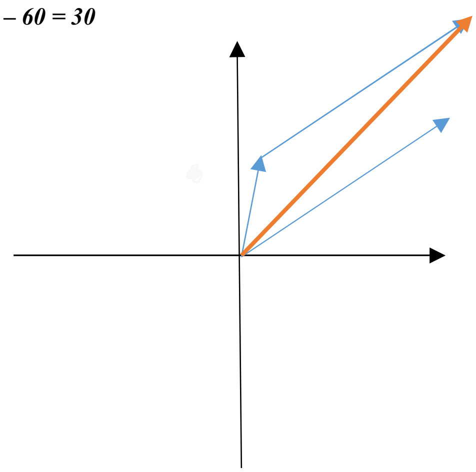
If $\mathbf{F} = a\mathbf{i} + b\mathbf{j}$, the direction angle, $\theta$, of $\mathbf{F}$ can also be found using

$$\tan\theta = \frac{b}{a}.$$

Ex. Two forces **F** and **G** of magnitude 30 and 60 pounds respectively act on an object. The direction of **F** is N10°E and the direction of **G** is N60°E. Find the magnitude and direction of the resultant force. Express the magnitude to the nearest hundredth of a pound and direction angle to the nearest tenth of a degree.

***Using directionality: N10°E means 10°E of due North From the positive x-axis, it would be 90 – 10 = 80***

***N60°E means 60°E of due North. From the positive x-axis, it would be 90 – 60 = 30***



$$F = \|F\| \cos\theta\, \boldsymbol{i} + \|F\| \sin\theta \boldsymbol{j} = 30\cos 80\boldsymbol{i} + 30\sin 80\boldsymbol{j}$$

$$F = 30\cos 80 + 30\sin 80\boldsymbol{j} \approx 5.21\boldsymbol{i} + 29.54\boldsymbol{j}$$

$$G = \|G\| \cos\theta\, \boldsymbol{i} + \|G\| \sin\theta \boldsymbol{j} = 60\cos 30\boldsymbol{i} + 60\sin 30\boldsymbol{j}$$

$$G = 60\cos 30\boldsymbol{i} + 60\sin 30\boldsymbol{j} \approx 51.96\boldsymbol{i} + 30\boldsymbol{j}$$

***The resultant force, V=F + G is***
*V = (5.21+51.96)**i** + (29.54+30)**j** = 57.17**i** + 59.54**j***

$$\|V\| = \sqrt{a^2 + b^2} = \sqrt{(57.17)^2 + (59.54)^2} \approx 82.54 \textit{ pounds}$$

$$\theta = \cos^{-1}\frac{a}{\|V\|} = \cos^{-1}\frac{57.17}{82.54} \approx 46.16°$$

**Homework: Day 1: Pg. 709 #3-45 (3's)**
**Day 2: Pg. 709-712 #2, 7, 10, 14, 20, 32, 38, 41-51o, 61, 64, 65, 71, 73, 75, 76**
**Day 3: Pg. 711-712 #79 – 86, 102-107, 113, 114, 116, 118**